An Adaptive Lexicographical Ordering of Color Mathematical Morphology

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Abstract. Though mathematical morphology can process the binary and grayscale image successfully, this theory can not extend to the color image directly. In color space, a vector represents a pixel, so in order to compare vectors, vectorial orderings must be defined first. In this paper, an adaptive lexicographical ordering is proposed. This approach takes advantage of the vectorial processing of the classical lexicographical ordering, but it can arrange the components' rank adaptively. In other words, this approach can select the first, second and third component automatically. First, a generalization of an adaptive lexicographical ordering is introduced, and the results of this approach's erosion, dilation, opening and closing are offered. Comparative application results on color noise reduction and edge detection are also provided.

Keywords: mathematical morphology, lexicographical ordering, color image, vectorial ordering

1 Introduction

Mathematical morphology (MM) is a theory and technique for the analysis and processing of geometrical structures, based on set theory, lattice theory, topology, and random functions, which is most commonly applied to digital images. MM was originally developed for binary images, and was later extended to grayscale functions and images. Now, many people pay attention to the color image based on MM. Unfortunately, the extension of the concepts of binary and grayscale morphology to color images is not straightforward.

There are two key issues in MM: morphological operators and structuring element (SE). The flat SE is always used. Morphological operators base on the concepts of the supremum and infimum for a grayscale image. For example, erosion (ε_b) and dilation (δ_b) of a grayscale image **f** by a flat SE b are expressed as following:

$$\varepsilon_b(\mathbf{f})(x) = \inf_{s\in b}[\mathbf{f}(x+s)] \tag{1}$$

$$\delta_b(\mathbf{f})(x) = \sup_{s\in b}[\mathbf{f}(x-s)] \tag{2}$$

where "inf" and "sup" denote the infimum and supremum, respectively. In grayscale image, only one scale value denotes a pixel. By ranking, the infimum and supremum can be obtained easily.

However, a color image is made up of multi-scale images, almost three-scale images. So one color pixel includes three scale values, which results in the fact that the method of grayscale MM should not be applied for color MM, that is to say, we can not rank the color pixels directly by the scale-based pixel ordering scheme used in the grayscale image morphology. Thus, we must redefine the color morphological operators.

In color image, the color pixel is treated at each pixel as a vector. Similar to (1) and (2), the vectorial erosion and dilation can be expressed by means of the vectorial extrema operators "$\sup_v$"and "$\inf_v$" as following:

$$\varepsilon_b(\mathbf{f})(x) = \inf_{v\, s\in b}[\mathbf{f}(x+s)] \tag{3}$$

$$\delta_b(\mathbf{f})(x) = \sup_{v\, s\in b}[\mathbf{f}(x-s)] \tag{4}$$

In order to obtain the infimum and supremum of the vectors, an appropriate ordering of vectors must be defined. But no unambiguous ordering exists for vector-valued images, that is to say, how to rank the vectors is by no means an easy thing. Therefore, vectorial ordering is important to color morphology, on which great research efforts had been focused.

In this paper, some classical ordering methods are presented in Section 2. And a new approach is proposed in Section 3. In Section 4, some applications and comparison are provided.

2 Vectorial orderings

The key point to extend MM to color images is to define a well-suited ordering relation. In the past few decades, a lot of effort has been put in engineering a way of ordering vectors, which can be classified into four groups: marginal ordering (M-ordering), reduced ordering (R-ordering), partial ordering (P-ordering), and conditional ordering (C-ordering) [1, 2, 3].

2.1 M-ordering:

A color image can be separated into a set of components, and then a univariate ordering is applied to each component, finally the processed components are merged to form the output image. Since data is ordering along each one of the components independently from others, this approach is also named as the componentwise ordering. Thus, in fact M-ordering is not a real vectorial ordering approach.

This approach is simple, and grayscale morphological methods can be employed directly. However, this approach has two major drawbacks. First, this approach produces the new color vectors which are not originally present in the input image, so it often introduces color artifacts into the output image [2]. Second, because each channel of the image is processed separately, the inter-component correlation is totally ignored, along with all information that could be potentially used in order to improve the quality of the result [1].

2.2 R-ordering:

The vectors are first reduced to scalar values and then ranked according to their scalar order. Typically, distance is employed to be scalar values. For instance, R-ordering orders three-component vectorial valued observations according to their distance from some reference vector. As a consequence, multivariate ordering is reduced to one-dimensional ordering. Hence, it is obvious that the output image would depend not only on the input image, but also on the distance measure and reference vector.

The distance measure can be the L_2 norm, the Mahalanobis distance, or any other distance measure. Ideally the reference vector should be the true value of the underlying vector that is to be estimated. In practice any suitable multivariate estimator of location (e.g. the arithmetic mean, the marginal or vector median) evaluated over a subset of the input data, can be used as reference vector. Moreover, this approach may lead to the existence of more than one extremes and, thus, introduce ambiguity in the resultant data [4]. However, R-ordering is easy to implement and the most natural for vector-valued observations [5, 6].

2.3 P-ordering:

This approach is based on the partition of the vectors into group, such that the groups can be distinguished with respect to rank or extremeness. This is computed by using convex-hull like sets [2]. It is generally geometric in nature and account well for the inter-relations between components [1]. However, this approach may also lead to the disadvantage of multiple extremes.

2.4 C-ordering:

The vectors are ordering by means of some of their marginal components, and selected sequentially according to different conditions. However, the components not participating in the comparison process are listed according to the position of their ranked counterparts. Hence, the ordering of the vectors is conditioned on the particular marginal set of ranked components. In other words, C-ordering restricts the ordering process to only one or more components of the given vectors, while the others are conditioned on them. Thus, this approach is suitable for situations where certain component is more privileged than others [1].

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Lexicographical ordering is one kind of C-ordering, which is relative popularity. The lexicographical method is the order in which words are arranged in dictionaries: first, the order is decided by a component followed by a second, and finally by a third. Let $\mathbf{x} = \{x_1, x_2, x_3\}$ and $\mathbf{y} = \{y_1, y_2, y_3\}$ be two arbitrary vectors, an example of lexicographical ordering may be

$$\mathbf{x} < \mathbf{y} \text{ if } \begin{cases} x_1 < y_1 \text{ or} \\ x_1 = y_1 \text{ and } x_2 < y_2 \text{ or} \\ x_1 = y_1 \text{ and } x_2 = y_2 \text{ and } x_3 < y_3 \end{cases} \tag{5}$$

Due mainly to its theoretical properties, the lexicographical ordering is widely used, since it makes it possible to totally order the underlying pixel data while also providing unique extreme [7]. However, lexicographical ordering has a major drawback, which is limited in practice to cases where the first component of the compared vectors holds the majority of the data of interest compared to the rest of the image components [8].We would like to stress that the class/style files and the template should not be manipulated and that the guidelines regarding font sizes and format should be adhered to. This is to ensure that the end product is as homogeneous as possible.

Fig. 1. The original image

(a) RGB (b) GBR (c) BRG

Fig. 2. Erosion using classical lexicographical ordering

Now, an example is provided that the first component is very important for lexicographical ordering. Here RGB color space is used, and the original image is applied by vectorial erosion and dilation respectively with 7× 7 square flat SE. When we permute the channels as RGB, GBR and BRG respectively, the results have some differences. Fig.1, Fig.2 and Fig.3 are an original image, erosion results and dilation results, respectively.

(a) RGB (b) GBR (c) BRG

Fig. 3. Dilation using classical lexicographical ordering

Consequently, this situation leads to an inefficient exploitation of the inter-component relations. In order to overcome this drawback, α-lexicographical ordering was proposed [7,8].

3 An adaptive lexicographical ordering

As stated above, although lexicographical ordering is widely used in color morphology, the result depends heavily on the order of the image components. If we have a priori knowledge that one component is priority over other components, we should give first rank to this component. However, we can not have this priori knowledge in advance, which limits in practice for lexicographical ordering.

In lexicographical ordering, the result of the output image is determined mostly by the first component, and least by the last component. In RGB color space, we consider that the maximum value component of three components determines mostly the nature of image, so this component is selected to be the first component. In the same way, the minimum value component is the last component. Hence, in this approach, before the ordering is arranged, we must obtain the maximum, median and minimum of three components. In order to compute these values, not only the whole image is reflected, but also the SE.

Let **f** be a color image that **f**: R^2 -> RGB, and b a flat SE. Each component is respectively computed as following:

$$\mathrm{com}_{\mathrm{R}} = \beta \frac{\sum_{p\in R} \mathbf{f}_{\mathrm{R}}(p)}{R} + (1-\beta)\frac{\sum_{p\in b} \mathbf{f}_{\mathrm{R}}(p)}{b} \quad \textbf{(6)}$$

$$\mathrm{com}_{\mathrm{G}} = \beta \frac{\sum_{p\in R} \mathbf{f}_{\mathrm{G}}(p)}{R} + (1-\beta)\frac{\sum_{p\in b} \mathbf{f}_{\mathrm{G}}(p)}{b} \quad \textbf{(7)}$$

$$\mathrm{com}_{\mathrm{B}} = \beta \frac{\sum_{p\in R} \mathbf{f}_{\mathrm{B}}(p)}{R} + (1-\beta)\frac{\sum_{p\in b} \mathbf{f}_{\mathrm{B}}(p)}{b} \quad \textbf{(8)}$$

where $\mathbf{f}_{\mathrm{R}}(p)$, $\mathbf{f}_{\mathrm{G}}(p)$ and $\mathbf{f}_{\mathrm{B}}(p)$ denote the red component, green component, and blue component, respectively. Now $\mathrm{com}_{\mathrm{R}}$, $\mathrm{com}_{\mathrm{G}}$ and $\mathrm{com}_{\mathrm{B}}$ are ranked to determine which is maximum, median or minimum. And, the maximum is arranged as the first component, and minimum is the last component. Then, based on (5), the proposed lexicographical ordering is obtained. Here, β ($0 \leq \beta \leq 1$) is a weighted coefficient, which can influence the visual effects of the output image. After deep tests, we have found that the output image should be poor visual effect if β is too small.

Fig. 4 and Fig. 5 are the examples of the proposed approach. As compared with the classical lexicographical ordering, we still use the Fig.1 as the original image, which is applied by vectorial erosion and dilation respectively with 7×7 square flat SE. Here $\beta = 0.8$.

Fig. 4 and Fig. 5 are different from Fig. 2 and Fig. 3. Moreover, the proposed approach is not only adaptive to determine which is the first components, but also has a good visual effect.

Basing on erosion and dilation, opening and closing are defined, which are other important morphological operators. An opening is an erosion followed by a dilation, and a closing is a dilation followed by an erosion:

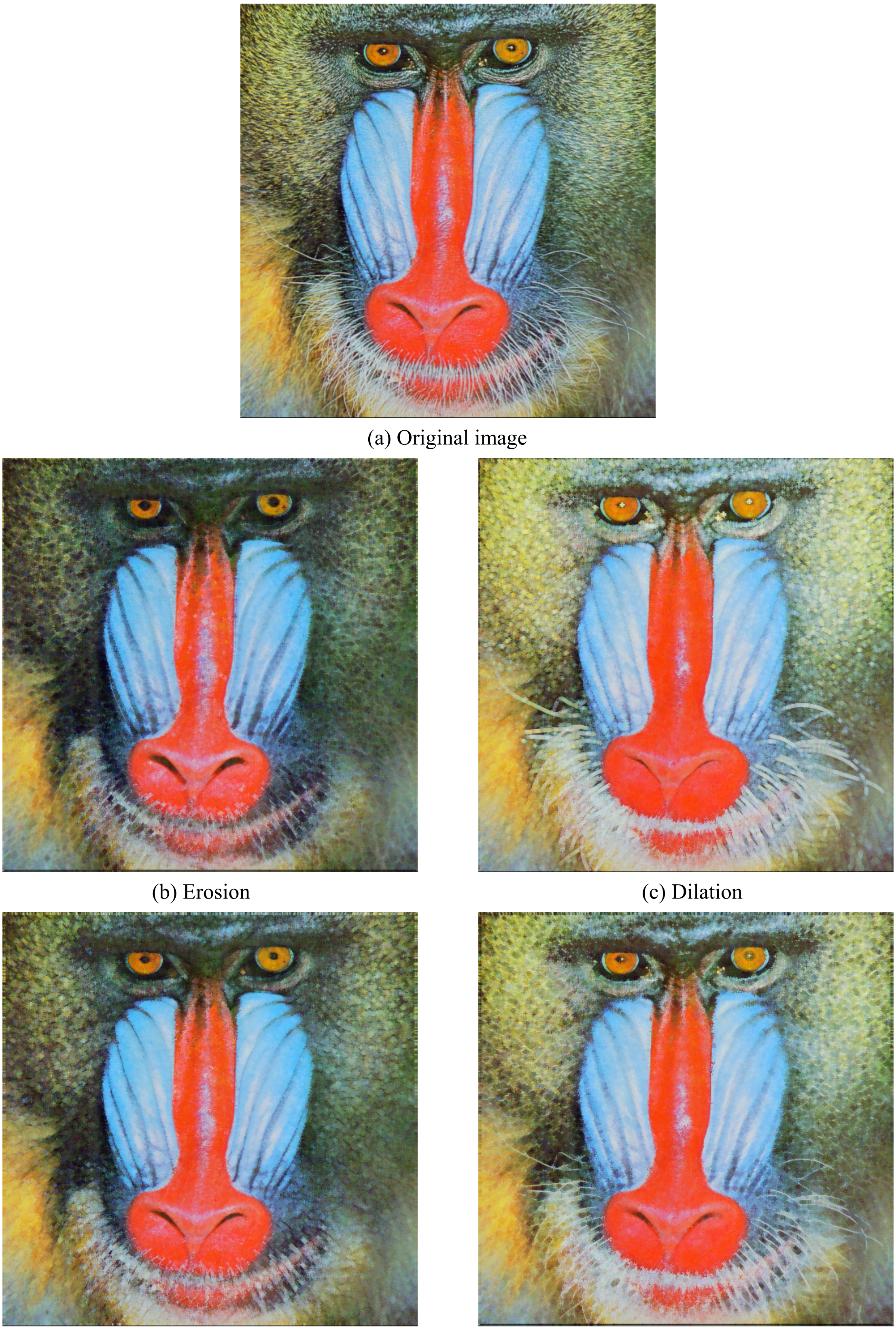
$$\gamma_b(\mathbf{f}) = \delta_b[\varepsilon_b(\mathbf{f})] \quad \textbf{(9)}$$

$$\phi_b(\mathbf{f}) = \varepsilon_b[\delta_b(\mathbf{f})] \quad \textbf{(10)}$$

Figure 4. Erosion by the proposed approach

Figure 5. Dilation by the proposed approach

Fig.6 is another actual example using the proposed approach with 5×5 cross flat SE. Here $\beta = 0.7$.



(a) Original image

(b) Erosion

(c) Dilation

(d) Opening

(e) Closing

Fig. 6. Another example by the proposed approach

Although the proposed approach presents the new idea by which the first component can be selected adaptively, it still uses the vectorial comparison equation (5) of the classical lexicographical ordering. Thus, the proposed approach must satisfy the morphological properties, such as extensive and anti-extensive property, increasing property, idempotence property, translation-invariance property, duality property, and so on.

4 Applications

The proposed approach can select the first component based on lexicographical ordering adaptively, which is superior to the classical lexicographical ordering.

In RGB color space, we can sort these three components as RGB, RBG, GRB, GBR, BRG, and BGR. In the morphological image process by the classical lexicographical ordering, the results must be different by these six permutations. In this section, we compare the proposed approach with the classical lexicographical ordering by noise reduce and edge detection. In the proposed approach, the weighted coefficient β is needed. We also change β to show the different results.

4.1 Noise reduction

The first application is noise reduction. As a quantitative measure, the normalized mean squared error (NMSE) is used:

$$\text{NMSE} = \frac{\sum_{i=1}^{N}\sum_{j=1}^{M}\left\|\mathbf{f}(i,j)-\mathbf{f}'(i,j)\right\|^2}{\sum_{i=1}^{N}\sum_{j=1}^{M}\left\|\mathbf{f}(i,j)\right\|^2} \tag{11}$$

where N and M represent the image dimensions while $\mathbf{f}(i, j)$ and $\mathbf{f}'(i, j)$ denote the vectorial pixels at position (i, j) for the original and filtered images, respectively. $\| \ \|$ represents the Euclidean norm.

The tests have been repeated with a few of RGB color images of various contents. However, we only present the results obtained for Fig. 7. We have also tested some noise distributions, but we show only Gaussian noise ($\sigma = 0.15$, $\rho = 0$) and salt & pepper noise (3%).

Fig. 7. Original image for noise reduction

In order to reduce the noise by MM, the common method is OCCO [1], which is defined as the pixelwise average of open-close and close-open:

$$\text{OCCO}_b(\mathbf{f}) = \frac{1}{2}\gamma_b[\phi_b(\mathbf{f})] + \frac{1}{2}\phi_b[\gamma_b(\mathbf{f})] \tag{12}$$

The SE b is chosen as a square of size 3×3.

The NMSE results are showed in Table 1 and Table 2. From Table 1 and Table 2, we can know that by choosing different rank, the NMSE results of the classical lexicographical ordering are different. However, as long as the weighted coefficient β has been fixed, the NMSE results of the proposed approach are determined. In addition, along with the increase of β, the NMSE results are decrease. In other words, the larger β is, the better the effects of noise reduction is. And the results of $\beta = 0.9$ and $\beta = 1$ are almost same. Moreover, when β is larger than a certain value, the proposed approach is better than some ranks of the classical lexicographical ordering in noise reduction.

Table 1. NMSE results of Gaussian noise reduction

	The classical lexicographical ordering						The proposed approach			
	RGB	RBG	GBR	GRB	BRG	BGR	$\beta = 0.6$	$\beta = 0.8$	$\beta = 0.9$	$\beta = 1$
NMSE	0.2098	0.2080	0.2207	0.2102	0.2122	0.2250	0.2611	0.2301	0.2169	0.2170

Table 2. NMSE results of salt & pepper noise reduction

	The classical lexicographical ordering						The proposed approach			
	RGB	RBG	GBR	GRB	BRG	BGR	$\beta = 0.6$	$\beta = 0.8$	$\beta = 0.9$	$\beta = 1$
NMSE	0.0303	0.0304	0.0296	0.0296	0.0294	0.0296	0.0487	0.0299	0.0294	0.0295

4.2 Edge detection

The classical definition of the morphological gradient for a grayscale image **f** is given by [9]:

$$\nabla \mathbf{f} = \delta_b(\mathbf{f}) - \varepsilon_b(\mathbf{f}) \tag{13}$$

When **f** is a color image, $\delta_b(\mathbf{f})$ and $\varepsilon_b(\mathbf{f})$ are color vectors, and there are several definitions of the color morphological gradient. Here we apply the following formula [10]:

$$\nabla \mathbf{f} = \left\| \delta_b(\mathbf{f}) - \varepsilon_b(\mathbf{f}) \right\| \tag{14}$$

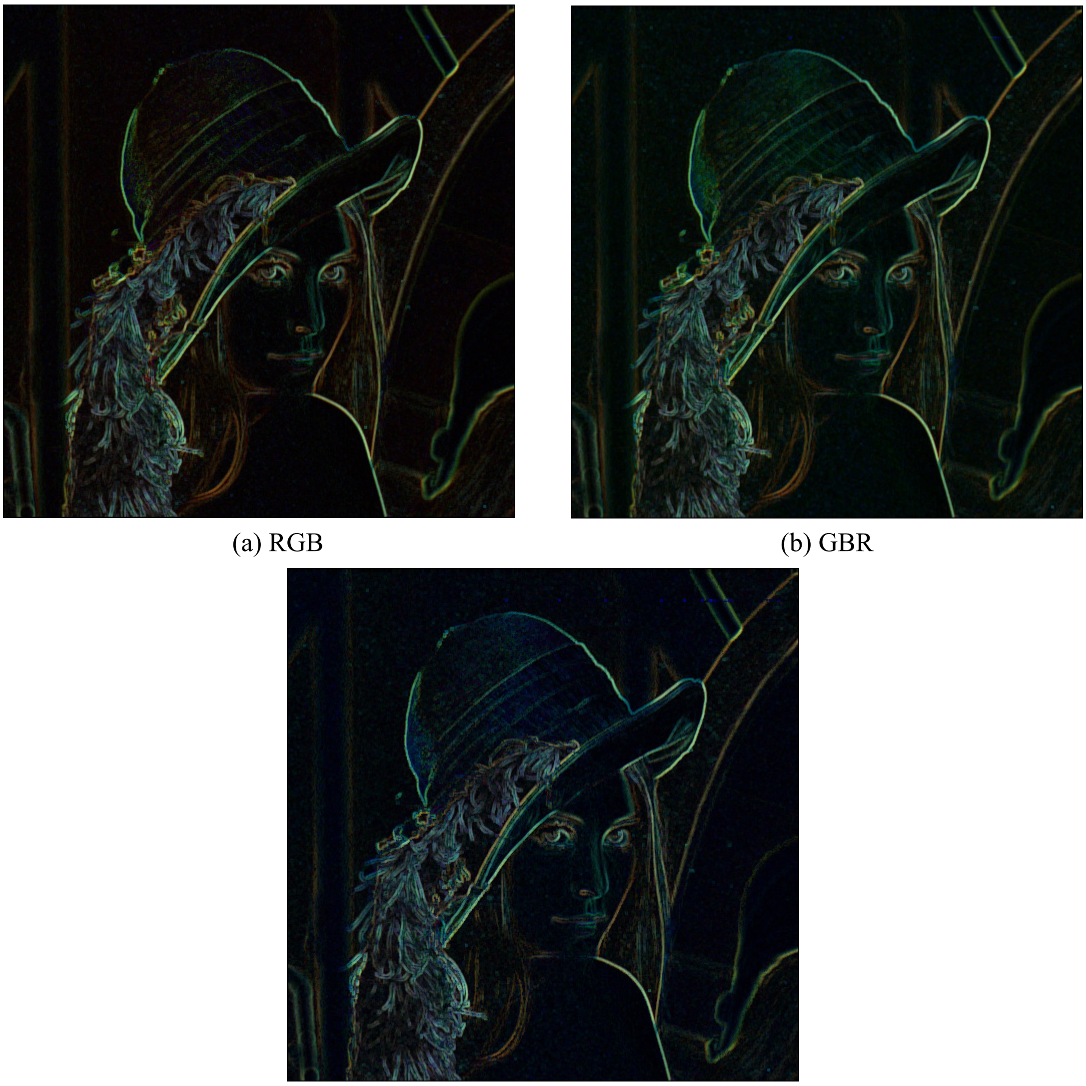
where || || represents the Euclidean norm.

Fig. 8. Original image for edge detection

Fig. 8 is the tested original image. Fig. 9 and Fig. 10 are the edge image by the classical lexicographical ordering and the proposed approach, respectively. Here, we only present the results of RGB, GBR and BRG.

Because of the characteristic of the classical lexicographical ordering that the first component has a pivotal role, from Fig. 9, we can see that the RGB edge image has the more red edges, the GBR edge image has the more green edges image, and the BRG edge image has the more blue edges. Thus the RGB edge image appears more red than other edge images, the GBR edge image appears more green than others, and the BRG edge image appears more blue than others. Because the color of the whole original image is inclined to be red, the edge image by the proposed approach and the RGB edge image of the classical lexicographical ordering are more similar.

In Fig.10, the coefficient β is 0.6. After deep tests, we have found that by different β, the edge images make little or no difference.



(a) RGB (b) GBR

(c) BRG

Fig. 9. Edge detection of the classical lexicographical ordering

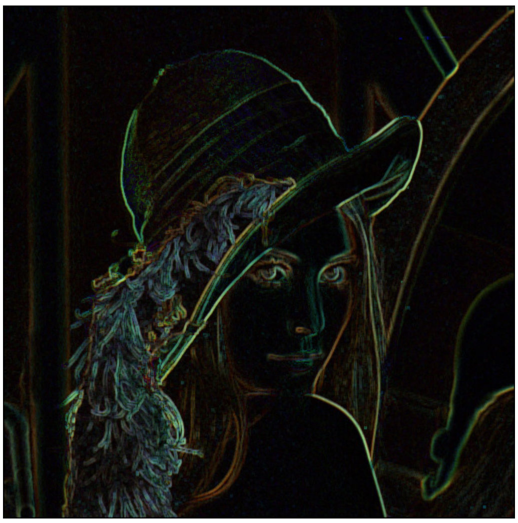


Fig. 10. Edge detection of the proposed approach

5. Conclusion

Mathematical morphology is an important tool for image processing. In the past, mathematical morphology had been already successfully applied to the binary image and grayscale image. However, in order that mathematical morphology can process the color image, vectorial ordering should be defined. M-ordering, R-ordering, P-ordering, and C-ordering are four kinds of vectorial orderings, but each of these orderings has both strong and weak points.

Lexicographical ordering is one kind of the C-ordering, which is widely used in the color morphology. However, in lexicographical ordering, the first component is so important that it can influence the output image.

In this paper, an adaptive lexicographical ordering is proposed. This approach can arrange the components' rank automatically. From the applications of noise reduction and edge detection, this approach is an effective method, which not only has a good visual effect, but also can process the color image as well as other vectorial orderings.

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