Reliability Envelope Analysis

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Received 19 July 2014; Revised 26 October 2014; Accepted 3 December 2014

Abstract. The modern computer and network system, is composed of tens of thousands of components. Any faults in components can affect reliability of the whole system. In the process of running, elements will gradually age and failure rate gradually increase. However, the rate at which these elements age is not synchronous. Thus, in which cases should the elements be upgraded or replaced and in which cases should the elements whose performance is still relatively good be retained, are hard job to handle. In this paper, the author attempts to make an extreme analysis on the reliability of parallel system with N element obeying exponential distribution to improve the system update.

Keywords: reliability, parallel system, exponential distribution, system update

1 Introduction

1.1 Reliability Introduction

Reliability theory was a newly emerged interdisciplinary subject in 1960s and analyzes the probability of random events which characterize the specified function of product. It is established on the basis of probability theory, which is an area of study focused on machine maintenance[1]. With the development of reliability theory, it gradually needs much frontier knowledge and tools in mathematics, while reliability mathematics has laid a food foundation for it. In practical reliability problems, the mathematics used can be divided into two categories: probability model and statistical model. Probability model infers the reliability indices of system on the basis of system structure and life distribution of components; while statistical model evaluates and tests the life of components or system on the basis of observed data. In the paper, statistical model is applied. Currently, the main researches focus on the reliability indices of system and optimal detecting time which is determined by reliability indices to avoid the occurrence of faults and reduce the losses caused by faults, such as literature[2],[3],[4]. In this paper, the author attempts to make an extreme analysis on the reliability of parallel system with N element obeying exponential distribution to improve the system update.

1.2 The Definition of the Main Indicators of Reliability

(1) Reliability

The definition of reliability *R*(*t*) [5]: it is the probability that product completes the required function under the specified conditions and within the prescribed time.

If the life distribution of product is $F(t)$, $t > 0$, the reliability $R(t) = P(T \geq t) = 1 - F(t)$ .This is a function of time(*t*), so it can be called as reliability function. To the components obeying exponential distribution λ , its reliability is $e^{-\lambda t}, t \geq 0$.

(2) Failure rate

Failure rate $\lambda(t)$: It is the probability of occurring failure in the unit of time after product has worked a period of time(t). According to reliability theory, $\lambda(t)=\dfrac{f(t)}{1-F(t)}$, when $t>0$, the failure rate of exponential distribution is constant λ.

(3) System parameter specification

A : represents normal working events of system.

A_i : represents normal working events of the element i.

λ_i : represents failure rate of the element i.

$R_s(t)$: represents system reliability, that is, $P(A)=R_s$.

$R_i(t)$: represents reliability of the element i, that is, $P(A_i)=R_i$.

Parallel system: It is a system consisting of n components. As long as one of these elements works, the system can work; only when all the units fail, the system would fail. The logical block diagram of parallel system with n element as follows (Figure 1).

According to the property of probability, the normal working probability of system $P(A)=P(\bigcup_{i=1}^{n} A_i)$ is as follows:

$$R_s(t)=1-\prod_{i=1}^{n}(1-R_i(t))=1-\prod_{i=1}^{n}(1-e^{-\lambda_i t}) \tag{1}$$

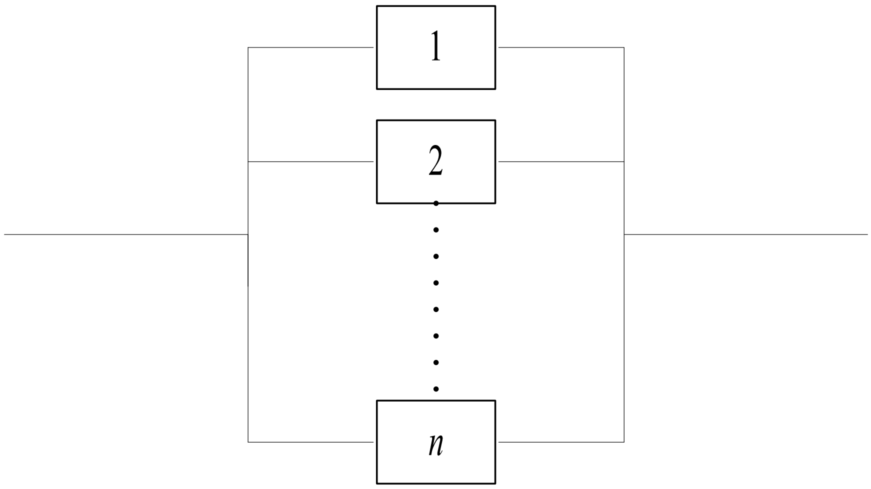

Fig. 1. The logical block diagram of parallel system

2 Model Analysis

2.1 Performance Constraint

According to the definition of parallel system reliability, when the overall reliability of N elements is fixed, the model is as follows:

$$\min R_s(t)=1-\prod_{i=1}^{n}(1-e^{-\lambda_i t}) \tag{2}$$

$$s.t.\sum_{i=1}^{n}\lambda_i=c,\lambda_i>0$$

where C is the sum of failure rate.

This model is equivalent to the following one:

$$\max \quad \left(\prod_{i=1}^{n} (1-e^{-\lambda_i t})\right) \tag{3}$$
$$s.t. \sum_{i=1}^{n} \lambda_i = c, \lambda_i > 0$$

If multivariable differential calculus and lagrangian multiplier are adopted here, the process of calculating conditional extremum will be complex. Thus, the author attempts to adopt Cauchy inequality here and conclude as follows:

$$\prod_{i=1}^{n} (1-e^{-\lambda_i t}) \le \left(\frac{\sum_{i=1}^{n} (1-e^{-\lambda_i t})}{n}\right)^n = \left(1-\frac{\sum_{i=1}^{n} e^{-\lambda_i t}}{n}\right)^n \tag{4}$$

Using the Cauchy inequality once again, the following can be concluded:

$$\left(1-\frac{\sum_{i=1}^{n} e^{-\lambda_i t}}{n}\right)^n \le \left(1-\sqrt[n]{\prod_{i=1}^{n} e^{-\lambda_i t}}\right)^n = \left(1-e^{-\frac{ct}{n}}\right)^n \tag{5}$$

The two inequalities above can be turned into equal, but this is only true when $\lambda_1 = \lambda_2 \cdots = \lambda_n = \frac{c}{n}$. Thus, when failure rate of all components are equal, the reliability of system would reach minimum value. In other words, when the sum of failure rate is certain, the reliability of selecting products with the same quality cannot be better than the reliability of choosing a relatively poor one from a good parallel product.

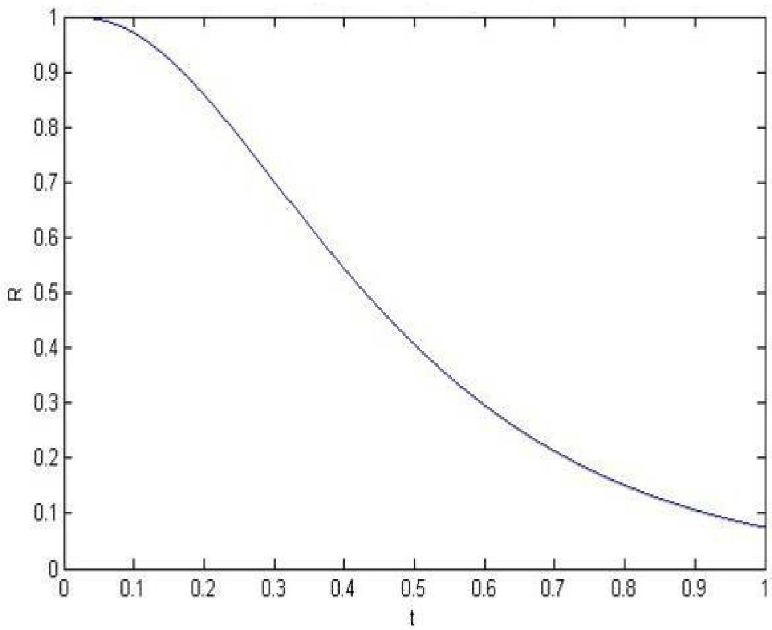

Fig. 2. The envelope curve of the lowest reliability

In practice, in the case that n elements in parallel are all from the parent which is obeying parameter λ , the abrasion and depreciation of components is growing over time, so the failure rate is in an increasing trend, thus c is increasing. Now assuming c is the linear function of t, the envelope curve of the lowest reliability in accordance with the time can be concluded as follows (Figure 2):

$$R_{\min}(t) = 1 - (1 - e^{-\frac{at^2+bt}{n}})^n, a < 0, b > 0 \quad \textbf{(6)}$$

wherein *a* represents wear rate, *b* is saturation mode when it is never used.

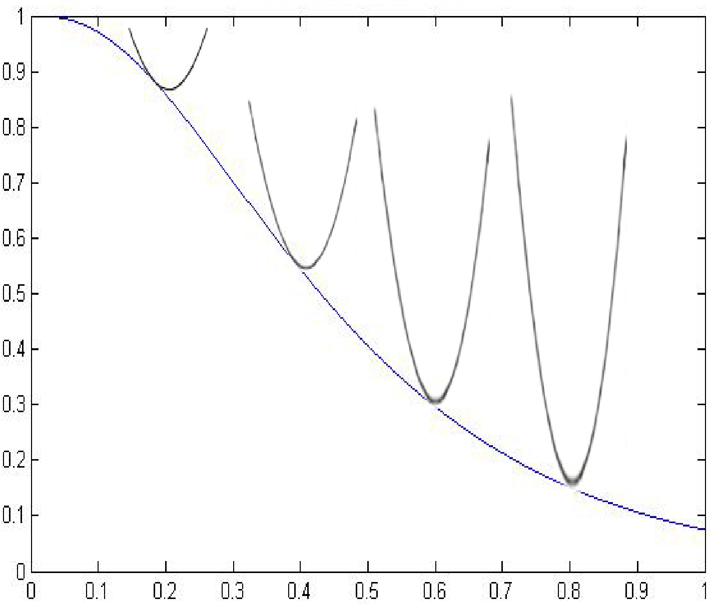

Fig. 3. System reliability and envelope curve

2.2 Relations between Cost and Failure Rate

Now the paper introduces the cost constraint, first we define the function of failure rate and corresponding costs, denoted as $\lambda = g(\mu)$, there into μ is expense, λ is still failure rate. Here we learn the definition margin of microeconomics, define the concept of marginal reliability [6]. Marginal reliability: the mount of decreasing failure rate arouse by increasing one unit of cost.

In actual, designers can choose relatively low failure rate product when he add budget, so $g'(u) < 0$. But with the increasing of cost, the amount of reducing failure rate is decreasing, thus there is $g''(u) > 0$. Next we denote the inverse function of $\lambda = g(\mu)$ as $u = f(\lambda)$, since $\lambda = g(\mu)$ and $u = f(\lambda)$ are mutually inverse function, so $g'(u) = \frac{1}{f'(\lambda)}$. According to the characteristics of the marginal failure rate, we can get $(\frac{1}{f'(\lambda)})' < 0$, that is $f''(\lambda) > 0$. Lastly, we define failure rate flexibility as the changing ratio of the failure rate divide changing ratio of cost, denote as E_λ. Its mathematical expression as following:

$$E_\lambda = \frac{\frac{du}{u}}{\frac{d\lambda}{\lambda}} = \frac{\lambda}{u}\frac{du}{d\lambda} \quad (7)$$

Failure rate flexibility can be divided into the following three types:

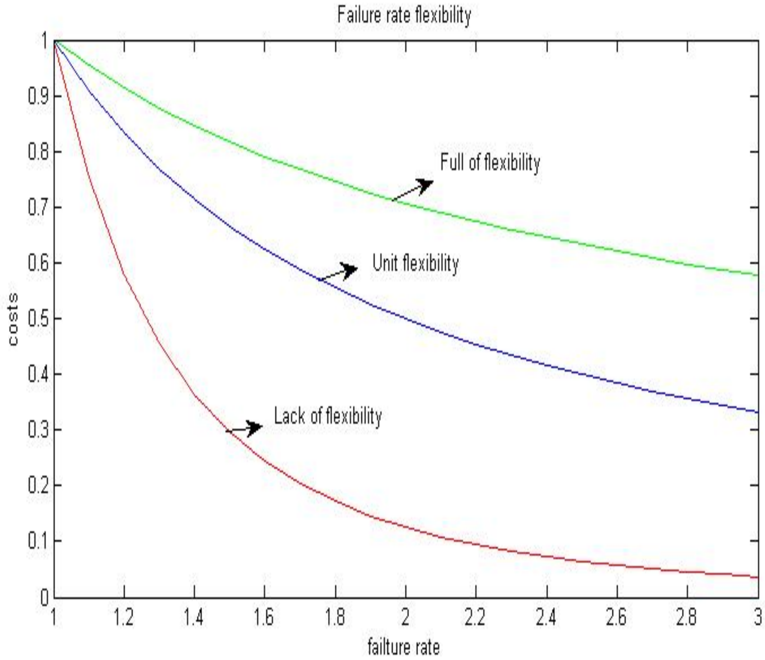

Fig. 4. Failure rate flexibility

(1) Lack of flexibility: $E_\lambda > 1$, for sophisticated originals, it requires a lot of R & D funding, therefore, the changing of failure rate ratio will cause large changing in the ratio of cost.

(2) Full of flexibility: $E_\lambda < 1$, for those who focus more on appearance innovation of product, its failure rate is full of flexibility because appearance innovation need less cost.

(3) Unit flexibility: $E_\lambda = 1$, for mature products and under full competition of market, the majority of products are such cases.

2.3 Costs and Constraint

By definition of parallel system reliability, under the performance and costs constraints, the model of *n* Element Obeying Exponential Distribution as below:

$$\min R_s(t) = 1 - \prod_{i=1}^{n} (1 - e^{-\lambda_i t})$$
$$s.t.\begin{cases} \sum_{i=1}^{n} \lambda_i \le c \\ \sum_{i=1}^{n} f(\lambda_i) \le u \\ \lambda_i > 0 \end{cases} \tag{8}$$

where c represents the performance constraint of the system, and u represents economic costs constraints of the system. Depending on the feature of $f(\lambda)$, firstly we consider the case of the failure rate flexible which is unit costs, based on formula (8): $E_\lambda = \dfrac{\frac{du}{u}}{\frac{d\lambda}{\lambda}} = \dfrac{\lambda}{u}\dfrac{du}{d\lambda} = 1$, learn by microeconomics, it represents the function of $u = 1/\lambda$. The model is replaced by the following:

$$\max R(t) = \prod_{i=1}^{n} (1 - e^{-\lambda_i t})$$
$$s.t. \begin{cases} \sum_{i=1}^{n} \lambda_i \le c \\ \sum_{i=1}^{n} \frac{1}{\lambda_i} \le u \\ \lambda_i > 0 \end{cases} \quad \textbf{(9)}$$

Firstly we analyse feasible solution space, according to the Cauchy-Schwarz inequality:

$$(\sum_{i=1}^{n}\lambda_i)*(\sum_{i=1}^{n}\frac{1}{\lambda_i}) \ge (\sum_{i=1}^{n}(\sqrt{\lambda_i}*\frac{1}{\sqrt{\lambda_i}}))^2 = n^2 \quad \textbf{(10)}$$

Thus we have:

$$u \ge n^2 / c \quad \textbf{(11)}$$

$$u \ge n^2 / c \Rightarrow n / u \le c / n \quad \textbf{(12)}$$

The inequalities above can be turned into equal, but this is only true when $\lambda_1 = \lambda_2 \cdots = \lambda_n = c / n$, formula (11) is a prerequisite for the existence of a feasible solution, also can be seen from the formula, in order to ensure the system have at least lowest performance, the least costs is $u = n^2 / c$, if $u < n^2 / c$, the costs is insufficient to support the current system of minimum performance. Having a feasible region of *u-c* relationship as shown in Figure 5, and known by Cauchy inequality, when $\lambda_1 = \lambda_2 \cdots = \lambda_n$, system reliability reaches a minimum, so whether is this solution feasible? When $\lambda_1 = \lambda_2 \cdots = \lambda_n = \lambda$, the minimal reliability of the system is the function of λ and t:

$$R_{\min}(\lambda) = 1 - \left(1 - e^{-\lambda t}\right)^n \quad \textbf{(12)}$$

the minimal reliability of the system is decreasing function of λ, and known by the inequality constraints:

$$n / \lambda \le u,\ n\lambda \le c \Rightarrow n / u \le \lambda \le c / n \quad \textbf{(13)}$$

λ can get to the maximum of c / n, therefore (3) can be reduced to the formula:

$$R_{\min} = 1 - (1 - e^{-\frac{c}{n}t})^n \quad \textbf{(14)}$$

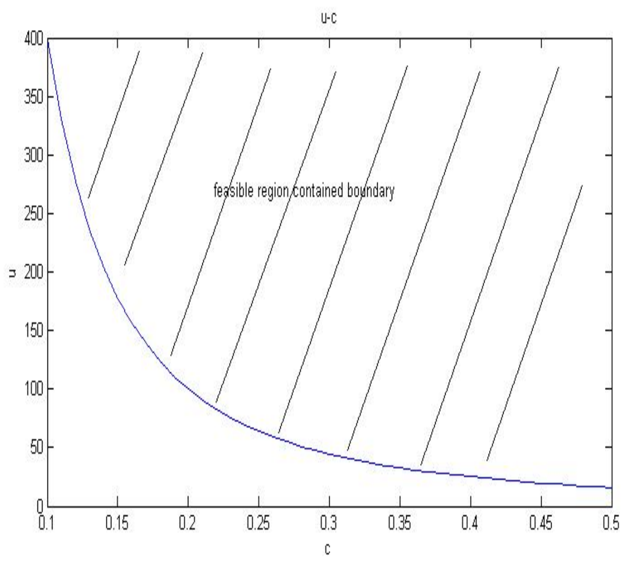

Fig. 5. Feasible solution space for u-c

The result is the same as only have performance constraint.

2.4 Using indifference curve analysis

Now we consider other type of flexibility, Firstly we analyse feasible solution space, according to the feature of $f(\lambda)$, take n = 2 for example, the feasible region of solution as shown in Figure 6. The size of the feasible region depends on the expense μ, and if μ is larger so that the size is larger, so it can provide more solution. When failure rate flexibility is unite, we use Cauchy-Schwarz inequality to get the least u which condition it has feasible solution space, but it doesn't give other situation as lack of flexibility or full of flexibility of failure rate. From formula (1), the feasible boundary is performance and economic constraints, and the feasible solution space is symmetry on coordinate axis

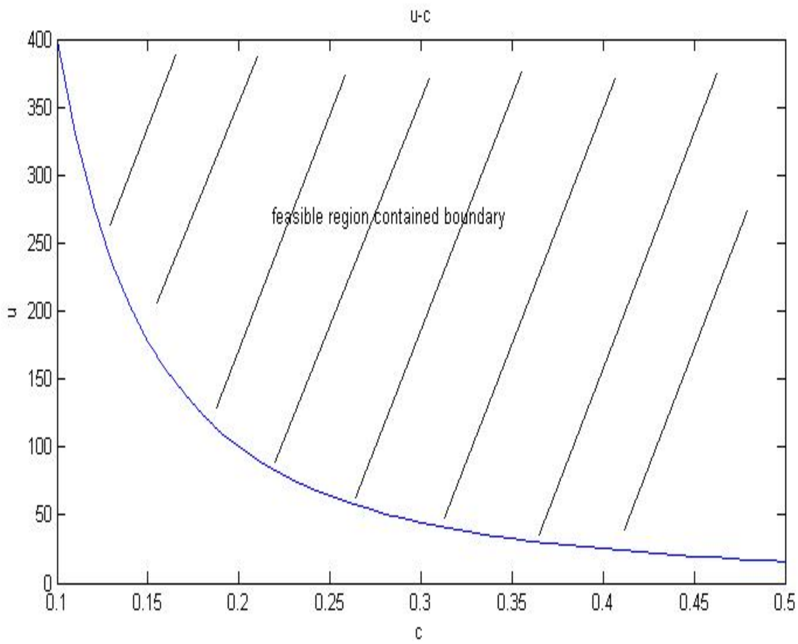

Fig. 6. Feasible solution space

Next, we study the reliability of indifference curve, consider t = 1 first, other cases can be transformed into the same analysis through setting $e^t = a$. The reliability of indifference curve shows in Figure 7. In the same of indifference curve, the reliability value is same, and the curve more closer from origin point, the reliability value is more greater, also can be seen that the marginal rate of substitution decreases with the increase of failure rate of one of element.

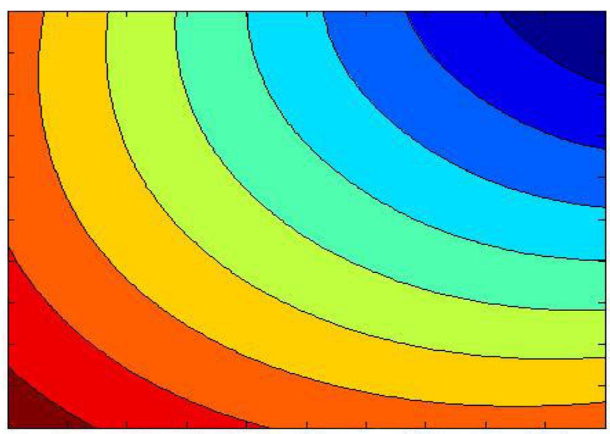

Fig. 7. Reliability indifference curve

Now we put feasible solution space and reliability indifference curve into one figure for analysing. As shown in figure 8,when reliability indifference curve and performance constraints boundary line is tangent, the system reliability reaches minimum, further more because of each curve is symmetry on the basis of y=x, so when $\lambda_1 = \lambda_2 \cdots = \lambda_n = c / n$, system reliability reaches minimum and it is

$$R_{\min} = 1-(1-e^{-\frac{c}{n}t})^n \tag{16}$$

When $n > 2$, the result is the same as formula (3) because of symmetry.

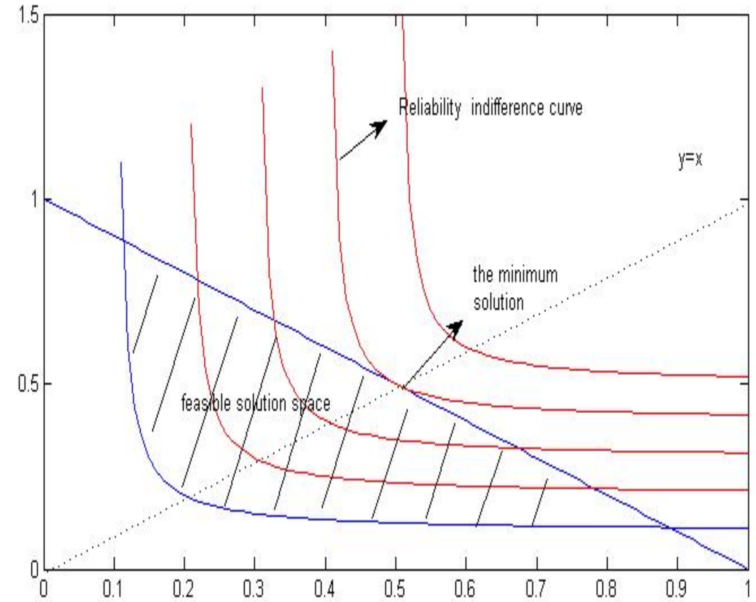

Fig. 8. Indifference analysis

3 Conclusion

On the condition that the sum of failure rate of system components is fixed, this paper attempts to make an extreme analysis on the reliability of parallel system with N element according to the Cauchy inequality, and put forward an meaningful conclusion and related theory as a guidance. When the failure rate of all components are equal, the system reliability can reach the minimum value, also under the performance and costs constraints, the result is same as only performance constraint, but it have least costs to support its feasible solution space which can't become empty set. Therefore, in practice, it is better to choose a relatively poor one from a good parallel product than select products with the same quality, because the reliability of the former is better than that of the latter. In the process of running, according to the actual situation of system, if the curve of failure rate is above the envelope curve, then all the components of the system have to be replaced; if the curve of failure rate is still relatively distant from the envelope curve, then the components with the highest failure rate are needed to replace according to the economy principle.

Acknowledgement

This research was partially supported by the the School of Mathematics and Computer Science of Fujian Normal University,by the the Institute of Innovative Information Industry of Fujian Normal University, by the The School of Ecomomic of Fujian Normal University.

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