Enumeration of 2D Lattice Paths with a Given Number of Turns

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Abstract. In this study, we address a lattice path enumeration problem. We derive a precise formula of unrestricted lattice paths, in a 2D integer rectangular lattice $\boldsymbol{L}(n_1, n_2)$ under the step set {<1, 0>, <0, 1>}, with a given number of *turns* totally but not NE-*turns* only. We give a combinatorial proof of the proposed formula, present results on some cases $\boldsymbol{L}(n_1, n_2)$ that are confirmed by an algorithm that deals with the generation of a two-item multiset $\{0^{n_1}, 1^{n_2}\}$ permutation, and show a graphical demonstration for a simple case $\boldsymbol{L}(3, 4)$. The proposed formula can be applied to a scheduling problem that deals with setup time between two types of machines, and can be extended to a 3D integer rectangular lattice under the step set {<1, 0, 0>, <0, 1, 0>, <0, 0, 1>}.

Keywords: lattice paths, enumeration, multiset permutation, scheduling, turn

1 Introduction

In this study we address a lattice path enumeration problem. There are several early works have been devoted to this problem [1, 8, 9, 11, 14, 15]. The number of papers pertaining to this problem has more than doubled each decade since 1960. Humphreys, in the paper entitled "*A history and a survey of lattice path enumeration*", gave a comprehensive and valuable introduction on this topic [4]. To describe a lattice path enumeration problem, four factors, namely the lattice, the set of steps a path may take, the restrictions imposed on path, and what path characteristics we are counting, have to be taken into account. First, a lattice is a subset of points in the space. The most common lattice is the 2D integral plane. That is, an integer rectangular lattice that has a horizontal x-axis and a vertical y-axis. Herein, $\boldsymbol{L}(n_1, n_2)$ denotes a 2D integer rectangular lattice. Second, a path is a route that usually starts at point (0, 0) then moves through a succession of steps, and finally ends at the target point (n_1, n_2). The most common step sets are {<1, 0>, <0, 1>} and {<1, 1>, <1, -1>}. Alternatively, there are three steps, more than three steps, and infinite step sets. In this study we adopt the step set {<1, 0>, <0, 1>}.That is, we consider lattice paths that are composed of unit horizontal and unit vertical steps in the positive direction. In other words, after leaving the starting point (0, 0), the paths only can apply unit steps eastward and units steps northward, but can change direction at any point (x, y) until it reaches the target point (n_1, n_2). Third, the lattice paths can be limited by the restrictions imposed on it. The term *restricted* lattice paths usually means that the paths are restricted by some boundaries. The boundary may be a line or a concatenation of multiple line segments. A path may have to stay above a boundary, below a boundary, or between two boundaries. The first and most common restriction is above or below the line $y = x$. The last, there are several path characteristics have been studied, including *encompassing area*, *turns*, *peak*, *hills* or *valleys*, *contacts* or *crosses*, *nonintersecting* and *osculating paths*. For further details, see [4]. In this study, we focus on the *turns* of a path.

There are various motivations behind the interest in the *turn* enumeration of lattice paths. Krattenthaler described three motivations, from probability, statistics, and commutative algebra, respectively [6]. He also showed the wide diversity of connections and applications in other domains like combinatorics, representation theory, and q-series. Krattenthaler counted lattice paths by keeping track of *turns* for paths between two parallel lines [6], and for pairs of paths [5]. Niederhausen gave explicit solutions for the enumeration of weighted left *turns* above a line [10]. All the papers mentioned above dealt with *restricted* lattice paths. In this study we focus on *unrestricted* lattice paths with no boundaries imposed and the path can move anywhere as long as the movement is in accordance with the step set. The motivation of our study comes from trying to solve a scheduling problem that deals with setup time between two types of machines.

To solve an enumeration problem, the goal would be a simple, explicit expression [12]. Our goal is to derive a simple, explicit formula of the *unrestricted* lattice paths, in a 2D integer rectangular lattice $\boldsymbol{L}(n_1, n_2)$ under the step set {<1, 0>, <0, 1>}, with a given number of *turns*. The solution method of this study is inspired by Krettenthaler's paper entitled "*The enumeration of lattice paths with respect to their number of turns*" [6]. By a *turn*, it means a vertex of a path where the direction of the path changes. Distinguishing between the two possible types of *turns*, Krettenthaler called a vertex of a path a North-East *turn* (NE-*turn*, for short) if it is the end point of a vertical step and at the same time the starting point of a horizontal step, and calls a vertex of a path an East-North *turn* (EN-*turn*, for short) if it is the end point of a horizontal step and at the same time the starting point of a vertical step [6].

Krettenthaler showed that if the answer for the enumeration of lattice paths with a given number of NE-*turns* is known, then solutions for several other enumeration problems can also be obtained. Therefore, it is sufficient to concentrate on the enumeration of lattice paths with given starting and end points, satisfying certain restrictions, and with a given number of NE-*turns*. The number of paths from point (a_1, a_2) to point (e_1, e_2) with exactly ℓ NE-turns, is immediately solved by

$$\binom{e_1 - a_1}{\ell}\binom{e_2 - a_2}{\ell}.$$

This solution comes from the observation that any path from point (a_1, a_2) to point (e_1, e_2) is uniquely determined by its NE-*turns*. The *x*-coordinates of the NE-*turns* can be chosen from $e_1 - a_1$ integers, while the *y*-coordinates can be chosen from $e_2 - a_2$ integers; and ℓ for each of those have also to be chosen [6].

However, in a 2D lattice $\boldsymbol{L}(n_1, n_2)$, a lattice path comprises NE-*turns* and EN-*turns* simultaneously and not NE-*turns* only. For example, in a 2D lattice $\boldsymbol{L}(3, 4)$, there are 18 paths with two NE-*turns*. More precisely, among these 18 paths, there are six paths actually with three *turns*, nine paths actually with four *turns*, and three paths actually with five *turns*. In addition to this, as mentioned earlier, our motivation comes from trying to solve a scheduling problem. Therefore, what we want to know is that how many paths with a given number of *turns*, totally but not NE-*turns* only.

The remainder of this paper is organized as follows. In section II, we propose the result, a precise formula of *unrestricted* lattice paths, in a 2D integer rectangular lattice $\boldsymbol{L}(n_1, n_2)$ under the step set {<1, 0>, <0, 1>}, with a given number of *turns* totally but not NE-*turns* only. In section III, we give a combinatorial proof of the proposed formula. In section IV, we present results on some cases $\boldsymbol{L}(n_1, n_2)$ that are confirmed by an algorithm that deals with the generation of a two-item multiset $\{0^{n_1}, 1^{n_2}\}$ permutation, and show a graphical demonstration for a simple case $\boldsymbol{L}(3, 4)$. Finally, conclusions are summarized.

2 Results

The main result of this study is Theorem 1. In what follows we will introduce two Lemmas to support Theorem 1. Note that, throughout the paper, all variables are positive integers.

***Definition* 1**. Let *t* denote the number of *turns* of a path that has *a* EN-*turns* and *b* NE-*turns*, that is, $t = a + b$.

***Lemma* 1**. $|a - b| = 0$ or $|a - b| = 1$.

Proof. It is clear that an NE-*turn* (or EN-*turn*) must be followed by an EN-*turn* (or NE-*turn*) unless the path already arrives the target point (n_1, n_2). ∎

***Lemma* 2**. If *t* is even $(2k)$, then $a = b = k$. If *t* is odd $(2k - 1)$, then $a = k$ and $b = k - 1$ when first step is eastward, or $a = k - 1$ and $b = k$ when first step is northward.

Proof. When *t* is even $(2k)$, by Lemma 1, it is clear that $|a - b| = 0$, thus we have $a = b = k$. When *t* is odd $(2k - 1)$, by Lemma 1, it is also clear that $|a - b| = 1$. Furthermore, if first step is eastward (i.e., the first *turn* is an EN-*turn*), then the last step must be northward (i.e., the last *turn* must be an EN-*turn*). Thus, we have $a = k$ and $b = k - 1$. Otherwise, we will have $a = b$, this is contrary to the fact that *t* is odd $(2k - 1)$. Similarly, if the first step is northward (i.e., the first *turn* is an NE-*turn*), then the last step must be eastward (i.e., the last *turn* must be an NE-*turn*). Thus, we have $a = k - 1$ and $b = k$. Otherwise, we will have $a = b$, this is contrary to the fact that *t* is odd $(2k - 1)$. ∎

Now, let $\boldsymbol{P}\,(n_1, n_2, t)$ denote the number of paths that with a given *t turns*. The goal of this study is to derive a concise formula of $\boldsymbol{P}\,(n_1, n_2, t)$. Obviously, $\boldsymbol{P}\,(n_1, n_2, 1) = 2$. The result is the following theorem.

***Theorem* 1**.

$$\boldsymbol{P}(n_1, n_2, 2k) = \binom{n_1 - 1}{k - 1}\binom{n_2 - 1}{k} + \binom{n_2 - 1}{k - 1}\binom{n_1 - 1}{k}, \tag{1}$$

and

$$\boldsymbol{P}(n_1, n_2, 2k - 1) = 2\binom{n_1 - 1}{k - 1}\binom{n_2 - 1}{k - 1}. \tag{2}$$

Proof. From the starting point (0, 0), if the first step is eastward then totally there are $(n_1 - 1)$ EN-*turns* that the path can choose and will not arrive directly to the target point (n_1, n_2). Similarly, from the starting point (0, 0), if the first step is northward then totally there are $(n_2 - 1)$ NE-*turns* that the path can choose and will not arrive directly to the target point (n_1, n_2).

For a path with even (2*k*) *turns*, by *Definition* 1 and *Lemma* 2, we know that it must be composed of *k* EN-*turns* and *k* NE-*turns*. Recall that an EN-*turn* must be followed by an NE-*turn* unless the path already arrives the target point (n_1, n_2).Therefore, if the first step is eastward (i.e., the first *turn* is an EN-*turn*), then the last step must be eastward (i.e., the last *turn* must be an NE-*turn*). Similarly, recall that an NE-*turn* must be followed by an EN-*turn* unless the path already arrives the target point (n_1, n_2). Therefore, if the first step is northward (i.e., the first *turn* is an NE-*turn*), then the last step must be northward (i.e., the last *turn* must be an EN-*turn*). Consequently, when the first step is eastward, what we need to do in the whole route are choose *k* EN-*turns* from $(n_1 - 1)$ EN-*turns* and choose $k-1$ NE-*turns* from $(n_2 - 1)$ NE-*turns,* and when the first step is northward, what we need to do in the whole route are choose *k* NE-*turns* from $(n_2 - 1)$ NE-*turns* and choose $k-1$ EN-*turn* from $(n_1 - 1)$ EN-*turns.* Since the first step is either northward or eastward, thus we have the following identity:

$$\boldsymbol{P}(n_1, n_2, 2k) = \binom{n_1 - 1}{k-1}\binom{n_2 - 1}{k} + \binom{n_2 - 1}{k-1}\binom{n_1 - 1}{k}.$$

For a path with odd $(2k-1)$ *turns*, by *Definition* 1 and *Lemma* 2, we know that it must be composed of *k* NE-*turns* and $k-1$ EN-*turns* when first step is northward, or that it must be composed of *k* EN-*turns* and $k-1$ NE-*turns* when first step is eastward. Therefore, if the first step is northward (i.e., the first *turn* is an NE-*turn*), then the last step must be eastward (i.e., the last *turn* must also be an NE-*turn*). Similarly, if the first step is eastward (i.e., the first *turn* is an EN-*turn*), then the last step must be northward (i.e., the last *turn* must also be an EN-*turn*). Consequently, when the first step is northward, what we need to do in the whole route are choose $k-1$ NE-*turns* from $(n_2 - 1)$ NE-*turns* and choose $k-1$ EN-*turns* from $(n_1 - 1)$ EN-*turns*, and when the first step is eastward, what we need to do in the whole route are choose $k-1$ EN-*turns* from $(n_1 - 1)$ EN-*turns* and choose $k-1$ NE-*turns* from $(n_2 - 1)$ NE-*turns*. Since the first step is either northward or eastward, thus we have the following identity:

$$\boldsymbol{P}(n_1, n_2, 2k-1) = 2\binom{n_1 - 1}{k-1}\binom{n_2 - 1}{k-1}. \ \blacksquare$$

3 Proof

In this section, we will give a combinatorial proof of the correctness of Theorem 1. The proof is straightforward. If we summing up over all values of *turns t* in Theorem 1, it should yields the total number of lattice paths from (0, 0) to (n_1, n_2), that is $\binom{n_1+n_2}{n_1}$. In what follows we first introduce a Lemma to support our proof.

***Lemma* 3**. For a 2D integer rectangular lattice $\boldsymbol{L}(n_1, n_2)$, assume that $n_1 \le n_2$, the minimum number of *turns* of a path is one and the maximum number of *turns* of a path is $2n_1 - 1$ if $n_1 = n_2$, or $2n_1$ if $n_1 < n_2$.

Proof. It is trivial that the minimum *turns* of a path is one. What we have to proof is for the maximum *turns* of a path.

Case 1: $n_1 = n_2$. Since there are totally $n_1 + n_2 - 1$ points to be passed through in a path from the starting point (0, 0) to the target point (n_1, n_2), the maximum *turns* of a path is $2n_1 - 1$ naturally, if the path makes a *turn* at each point (x, y).

Case 2: $n_1 < n_2$. We can view this 2D integer rectangular lattice $\boldsymbol{L}(n_1, n_2)$ as a 2D integer rectangular lattice $\boldsymbol{L}(n_1, n_1)$ concatenates another 2D integer rectangular lattice $\boldsymbol{L}(n_1, n_2\text{-}n_1)$. By case 1, we have a path from the starting point (0, 0) to the target point (n_1, n_1) with the maximum *turns* of $2n_1 - 1$. Now, from the first target point (n_1, n_1) to the second target (n_1, n_2) there are at most one more *turn* can be made, so the maximum *turns* of a path is $2n_1$. ■

Now, we can proceed to prove the correctness of Theorem 1.

***Theorem* 2**. For $n_1 < n_2$,

$$\sum_{t=1}^{2n_1} \boldsymbol{P}(n_1, n_2, t) = \binom{n_1 + n_2}{n_1}. \tag{3}$$

Proof.

$$\begin{aligned}\sum_{t=1}^{2n_1} \boldsymbol{P}(n_1, n_2, t) &= \sum_{t=1,3,\ldots,2n_1-1} \boldsymbol{P}(n_1, n_2, t) + \sum_{t=2,4,\ldots,2n_1} \boldsymbol{P}(n_1, n_2, t) \\ &= \sum_{k=1}^{n_1} \boldsymbol{P}(n_1, n_2, 2k-1) + \sum_{k=1}^{n_1} \boldsymbol{P}(n_1, n_2, 2k)\end{aligned}$$

$$=\sum_{k=1}^{n_1} 2\binom{n_1-1}{k-1}\binom{n_2-1}{k-1}+\binom{n_1-1}{k-1}\binom{n_2-1}{k}+\binom{n_2-1}{k-1}\binom{n_1-1}{k}$$

$$=\sum_{k=1}^{n_1}\binom{n_1-1}{k-1}\left[\binom{n_2-1}{k-1}+\binom{n_2-1}{k}\right]+\binom{n_2-1}{k-1}\left[\binom{n_1-1}{k-1}+\binom{n_1-1}{k}\right]. \quad (4)$$

By the most important binomial identity of all, the *addition formula* [3, p. 158],[1]we have

$$=\sum_{k=1}^{n_1}\binom{n_1-1}{k-1}\binom{n_2}{k}+\binom{n_2-1}{k-1}\binom{n_1}{k}$$

$$=\sum_{k=1}^{n_1}\binom{n_1-1}{k-1}\binom{n_2}{k}+\sum_{k=1}^{n_1}\binom{n_2-1}{k-1}\binom{n_1}{k}.$$

By the identity of sum of products of binominal coefficients [3, p. 169], we have

$$=\binom{n_1+n_2-1}{n_1}+\binom{n_1+n_2-1}{n_2}$$

$$=\binom{n_1+n_2-1}{n_1}+\binom{n_1+n_2-1}{n_1-1}$$

$$=\binom{n_1+n_2}{n_1}. \blacksquare$$

***Theorem* 3**. For $n_1 = n_2$,

$$\sum_{t=1}^{2n_1-1} \boldsymbol{P}(n_1, n_2, t)=\binom{n_1+n_2}{n_1}. \quad (5)$$

Proof.

$$\sum_{t=1}^{2n_1-1} \boldsymbol{P}(n_1, n_2, t)=\sum_{t=1,3,\ldots,2n_1-1} \boldsymbol{P}(n_1, n_2, t)+\sum_{t=2,4,\ldots,2(n_1-1)} \boldsymbol{P}(n_1, n_2, t)$$

$$=\sum_{k=1}^{n_1} \boldsymbol{P}(n_1, n_2, 2k-1)+\sum_{k=1}^{n_1-1} \boldsymbol{P}(n_1, n_2, 2k)$$

$$=\sum_{k=1}^{n_1} 2\binom{n_1-1}{k-1}\binom{n_2-1}{k-1}+\sum_{k=1}^{n_1-1}\binom{n_1-1}{k-1}\binom{n_2-1}{k}+\binom{n_2-1}{k-1}\binom{n_1-1}{k}$$

$$=\left\{\sum_{k=1}^{n_1-1} 2\binom{n_1-1}{k-1}\binom{n_2-1}{k-1}+\binom{n_1-1}{k-1}\binom{n_2-1}{k}+\binom{n_2-1}{k-1}\binom{n_1-1}{k}\right\}+2\binom{n_1-1}{n_1-1}\binom{n_2-1}{n_1-1}.$$

Since $n_1 = n_2$, we have

$$\binom{n_2-1}{n_1}=\binom{n_1-1}{n_1}=0.$$

So, we can substitute the term of

$$2\binom{n_1-1}{n_1-1}\binom{n_2-1}{n_1-1}$$

by the following terms of

$$2\binom{n_1-1}{n_1-1}\binom{n_2-1}{n_1-1}+\binom{n_1-1}{n_1-1}\binom{n_2-1}{n_1}+\binom{n_2-1}{n_1-1}\binom{n_1-1}{n_1}.$$

Thus, we can rewrite the last identity that with a brace as follows

$$=\sum_{k=1}^{n_1}\binom{n_1-1}{k-1}\left[\binom{n_2-1}{k-1}+\binom{n_2-1}{k}\right]+\binom{n_2-1}{k-1}\left[\binom{n_1-1}{k-1}+\binom{n_1-1}{k}\right].$$

Now, we obtain exact the same identity as (4) that occurred in the proof of Theorem 2. ■

4 Demonstration

[1]*Addition formula*: $\binom{r}{k}=\binom{r-1}{k}+\binom{r-1}{k-1}$, integer *k*.

In this section, we first present path distributions with respect to their *turns* on some cases of 2D integer rectangular lattice $\boldsymbol{L}(n_1, n_2)$ under the step set {<1, 0>, <0, 1>} in Table 1.

The correctness of these path distributions is confirmed by an algorithm that deals with the generation of multiset permutation [7]. It must be noted that permutation generation of a two-item multiset $\{0^{n_1}, 1^{n_2}\}$ can be viewed as path generation of a 2D lattice $\boldsymbol{L}(n_1, n_2)$. Then, for making a clear imagination, in what follows we show a graphical demonstration for a 2D integer rectangular lattice $\boldsymbol{L}(3, 4)$. By ***Theorem*** **1**, there are 2, 5, 12, 9, 6, and 1 path with 1, 2, 3, 4, 5, and 6 *turns* respectively. These paths are showed in Figs. 1 ~ 6 respectively.

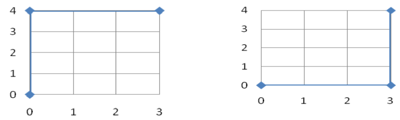

Fig. 1. 1NE (left diagram) vs. 1EN (right diagram)

Table 1. Path distributions with respect to their *turns* on $\boldsymbol{L}(n_1, n_2)$

turns	$\boldsymbol{n_1}$	**6**	**5**	**4**	**3**	**2**	**1**	**5**	**4**	**3**	**2**	**1**
	$\boldsymbol{n_2}$	**6**	**7**	**8**	**9**	**10**	**11**	**5**	**6**	**7**	**8**	**9**
1		2	2	2	2	2	2	2	2	2	2	2
2		10	10	10	10	10	10	8	8	8	8	8
3		50	48	42	32	18		32	30	24	14	
4		100	96	84	64	36		48	45	36	21	
5		200	180	126	56			72	60	30		
6		200	180	126	56			48	40	20		
7		200	160	70				32	20			
8		100	80	35				8	5			
9		50	30					2				
10		10	6									
11		2										
Total Paths		924	792	495	220	66	12	252	210	120	45	10

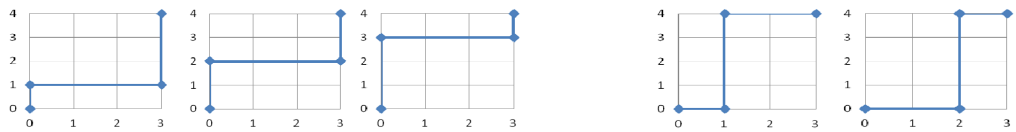

Fig. 2. 1NE+1EN (left 3 diagrams) vs. 1EN+1NE (right 2 diagrams)

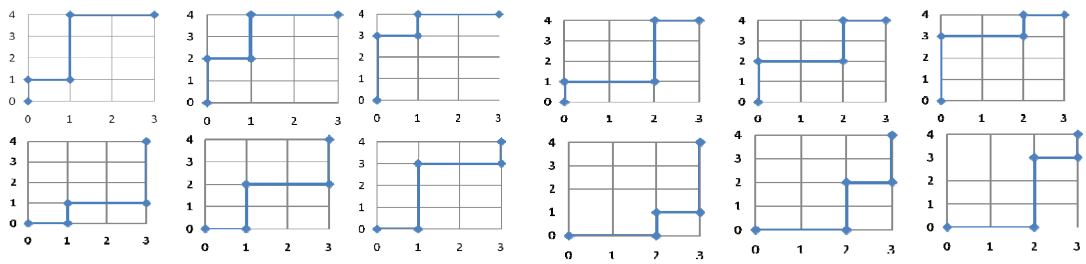

Fig. 3. 2NE+1EN (top 6 diagrams) vs. 2EN+1NE (bottom 6 diagrams)

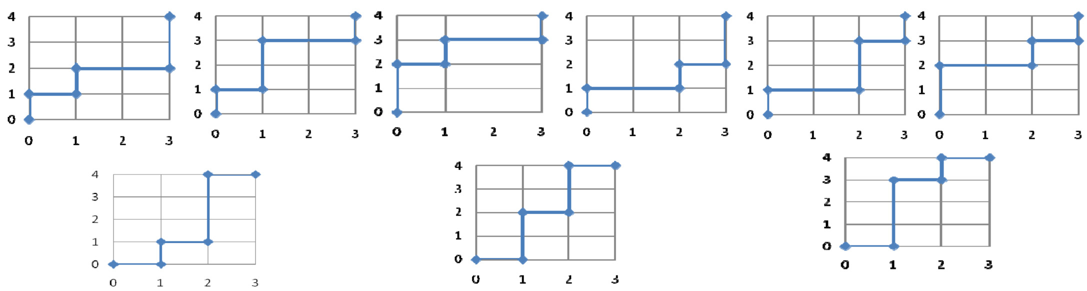

Fig. 4. 2NE+2EN (top 6 diagrams) vs. 2EN+2NE (bottom 3 diagrams)

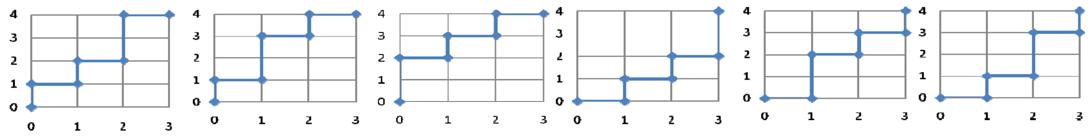

Fig. 5. 3NE+2EN (left 3 diagrams) vs. 3EN+2NE (right 3 diagrams)

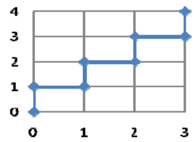

Fig. 6. 3NE+3EN

5 Conclusion

We derive a precise formula of *unrestricted* lattice paths, in a 2D integer rectangular lattice $\boldsymbol{L}(n_1, n_2)$ under the step set {<1, 0>, <0, 1>}, with respect to their number of *turns* totally but not NE-*turns* only. We give a combinatorial proof of the correctness of the proposed formula, present results on some cases $\boldsymbol{L}(n_1, n_2)$ that are confirmed by an algorithm that deals with the generation of a two-item multiset $\{0^{n_1}, 1^{n_2}\}$ permutation, and show a graphical demonstration for a simple case $\boldsymbol{L}(3, 4)$. It seems reasonable that the proposed formula can be extended to a 3D integer rectangular lattice under the step set {<1, 0, 0>, <0, 1, 0>, <0, 0, 1>}, although it is more or less complicated. The reason is twofold. First, we can make an analogy between lattice and building by perceiving that a 3D lattice $\boldsymbol{L}(n_1, n_2, n_3)$ is built on a 2D lattice $\boldsymbol{L}(n_1, n_2)$ ground floor then extended upward floor by floor to a total of n_3 floors. Second, in a 3D lattice $\boldsymbol{L}(n_1, n_2, n_3)$ a path with t *turns* can be obtained by extending upward a path in a 2D lattice $\boldsymbol{L}(n_1, n_2)$ ground floor with some fixed *turns* only, and in a 2D lattice $\boldsymbol{L}(n_1, n_2)$ a path with t *turns* can be extended upward to increase some fixed *turns*. Finally, it is worth to note that the proposed formula can be applied to a scheduling problem that deals with setup time between two types of machines.

In the scheduling problems, if the paths of jobs are fixed beforehand, and are the same for all jobs, it is called a *flow-shop*; on the other hand, if the paths of jobs are not given in advance, but chosen by a scheduler, it is called an *open-sho*p [13]. Let's consider an *open shop* scheduling of n machines of two types, A and B, with time delays. Assume that n_1 and n_2 be the number of machines of type A and type B respectively, and that $n = n_1 + n_2$. Delay time occurs when we transfer a job from a type A (or B) machine to a type B (or A) machine. Now, let $\pi_j = \pi_{j1}\pi_{j2} \ldots \pi_{jn}$ be a path of job j where π_{jk}, for all $k \in \{1, 2, \ldots, n\}$, can be a symbol of "A" or "B" that stands for the k^{th} operation of job j is processed by a machine of type A or B, respectively. To be a valid path, obviously, in a path π_j of a job j there must has n_1 A's and n_2 B's. For example, if $n_1 = 3$ and $n_2 = 4$ then $\pi_j = \text{AABBABB}$ and $\pi'_j = \text{ABBAABB}$ can be two feasible paths of a job j. It is trivial that there are totally $\frac{n!}{n_1! \times n_2!}$ different paths can be chosen for a job. If we let D_j stands for the total delay times needed for a job j to go through its route, then D_j is dependent on how many turns of the path that a job j is arranged. If the objective is to minimize $\sum_{j=1}^{j=J} D_j$, then a reasonable strategy is to choose an available feasible path with minimum turns as early as possible. To do that, we are confronted with two issues. The first one is to answer the question of what is the number of paths that possess a given number of turns; the second one is to design an algorithm that can generate all the paths in a non-decreasingly order of turns. We have finished the first issue and leave the second issue as a future work.

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