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# The Fuzzy Basis Function for the Adaptive Control with Robust Design

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Abstract. This paper presents the structure of model reference adaptive control based on fuzzy basis function. Initially, the techniques of fuzzy basis function are introduced to approach uncertain (or unknown) nonlinear systems. In addition, a heuristic adaptive algorithm, which is robust feature under the Lyapunov stability criterion is developed so that the output of the plant tracks a reference model output or follows another signal. As a result, the linear combination of the fuzzy basis function can approximate any nonlinear continuous function to any arbitrary accuracy that is substantiated by the Stone Weierstrass theorem. We use the linear combination coefficients to represent unknown system parameters and change the identification problem from identifying the original unknown parameters to the combination coefficients. The adaptive law based on tracking error and prediction error is presented and performance better can be expected than traditional MRAC in a time-varying nonlinear system. Furthermore, the effect of fuzzy logic approximation error will appear gradually when the number of the membership function is not enough. Therefore, a desired tracking performance cannot be guaranteed for a class of uncertain systems with external disturbances. In view of this, we impose a robust law in order to eliminate the influence of uncertain sources and approximation error. Two examples are given to illustrate the performance of the proposed method. Computer simulation results confirm that the proposed method not only satisfied tracking performance, but it also is vigorously robust.

Keywords: fuzzy basis function, Lyapunov stability criterion, model reference adaptive control, robust control

## 1 Introduction

The field of adaptive control began to develop in the 1950s. Subsequently, modern control theories, such as state space method, stability analysis, optimal control theory, and parameter identification, gradually developed in the next 20 years; thus assisting the development of adaptive control. Adaptive control is divided into 3 categories: gain scheduling, self-tuning regulators, and Model Reference Adaptive Controller (MRAC). MRAC was developed earliest and had been successfully applied to robotic arms, boat control, and aircraft control [1]. The main concept of MRAC is to promote the output of the system to track a reference model that has been constituted in advance, thereby accommodating the performance index of the system.

As to the design of MRAC with respect to the adaptive law, Massachusetts Institute of Technology (MIT) introduced the Local Parameter-Optimization theory, and thus this theory is known as the MIT method. This method did not incorporate stability into the design and, consequently, the system can diverge under certain circumstances. Accordingly, a design method based on the Lyapunov stability criterion was introduced to enhance stability, and how to find an appropriate Lyapunov function is important for the success of this method. After the Hyperstability Method was introduced, the MRAC structure based on the Hyperstability Method and the positive-definite property became increasingly important [2]. This method provides a systematic design method, thereby avoiding confusion in the selection of Lyapunov function. Furthermore, a number of studies on the combination of MRAC and modern control theory were made after the design of MRAC, incorporating the variable structure control theory.

The concepts of fuzzy sets and fuzzy inference have been extensively applied to a number of control and decision of adaptive systems [3-4]. Chen [5] successfully merged the concept of the Sliding Mode Control (SMC) into the design of adaptive law under the stability condition, with the Fuzzy Basis Function serving as a bridge between the mathematics of fuzzy theory and the adaptive control [6]. Here, adaptive fuzzy control system is divided into direct structure and indirect structure. Wang [7] designed an adaptive control law based on the fuzzy basis function for both structures. Subsequently, Lee and Yin [8] derived the MRAC law from the fuzzy basis function. This law is based on both tracking errors and estimated errors to activate adaptive law to adjust parameters online; and further, it regulates the membership function of input variables via the offline gradient method. Chen [9] incorporated $H^{\infty}$ robust control to eliminate the approximate errors due to fuzzy basis function. In addition to design for the structures of direct and indirect adaptive control, Chen also studied similar systems with measurable output (states must be determined by observer). Based on the discussion of $H^{\infty}$ optimization in Francis et al. [10], Zames [11] presented a criterion to assess the performance a control system, first using $H^{\infty}$ norm, and then applying $H^{\infty}$ robust control theory to practical systems.

Based on fuzzy basis function, we begin to develop the modeling of a system under MRAC; and design the adaptive law in compliance with Lyapunov stability. The robust control term is added to the control law to minimize the influence imposed by disturbance and errors, thereby ensuring that the system is not affected by external disturbances and fuzzy approximate errors. With the ideal combination of fuzzy, adaptive and robust control, the control architecture proposed in this paper guarantees perfect tracking performance.

## 2 Preliminary

### 2.1 Fuzzy Basis Function

Assume there are *n* input variables $(x_1, x_2, ....., x_n)$, a single output variable *y*, and *m* fuzzy rules:

$$R^l : IF\ x_1\ is\ A_1^l\ and\ x_2\ is\ A_2^l\ and ...and\ x_n\ is\ A_n^l\ , Then\ y\ is\ B^l \qquad l = 1, ..., m \tag{1}$$

When multiplication inference, single-point fuzzification and central average fuzzy solution are adapted, the output of fuzzy logic can be denoted as:

$$\begin{aligned} f(x) &= \underline{\xi}(\underline{x})\underline{\theta} = \frac{\sum_{l=1}^{m} B^l \prod_{j=1}^{n} \mu_{A_j^l}(x_j)}{\sum_{l=1}^{m} \prod_{j=1}^{n} \mu_{A_j^l}(x_j)} \\ \xi^l(\underline{x}) &= \frac{\prod_{j=1}^{n} \mu_{A_j^l}(x_j)}{\sum_{l=1}^{m} \prod_{j=1}^{n} \mu_{A_j^l}(x_j)} \end{aligned} \tag{2}$$

where $\underline{\xi}(\underline{x}) = [\xi^1(\underline{x}), \xi^2(\underline{x}), ..., \xi^m(\underline{x})]^T$ is referred to as the fuzzy basis function; and $\underline{\theta} = [\theta_1, \theta_2, ..., \theta_m]^T$ is referred as the adjustable parameters, thereby allowing $f(x)$ to approximate any continuous functions. In the function, $\mu_{A_j^l}$ is the membership function corresponding to $j^{th}$ input variable and $l^{th}$ rule.

### 2.2 System Description

Firstly, we consider the following $n$ order differential equation of a nonlinear uncertainty system:

$$y^{(n)} + a_{n-1}y^{(n-1)} + a_{n-2}y^{(n-2)} + ..... + a_0 y = u \tag{3}$$

where $y$ and $u$ represent the output and input variable of the system, respectively; the system parameter $a_i$ is probably the time-varying function in the other state variables; and in addition the reference model is described as:

$$y_m^{(n)} + \alpha_{n-1}y_m^{(n-1)} + \alpha_{n-2}y_m^{(n-2)} + ..... + \alpha_0 y_m = r \tag{1}$$

The parameters in the system are defined as follows:
$e = y - y_m$ denotes tracking errors; $\underline{a} = (a_n, a_{n-1}, ....., a_0)^T$ denotes system parameters; $\underline{\alpha} = (\alpha_n, \alpha_{n-1}, ....., \alpha_0)^T$ denotes the parameters of reference model; $\hat{\underline{a}}$ denotes estimated parameters; $\tilde{\underline{a}} = \hat{\underline{a}} - \underline{a}$ denotes the estimated errors; $\underline{\theta} = (\underline{\theta}_n, \underline{\theta}_{n-1}, ....., \underline{\theta}_0)$ denotes linear combined parameters; $\hat{\underline{\theta}}$ denotes estimated combined parameter; and $\tilde{\underline{\theta}} = \underline{\theta} - \hat{\underline{\theta}}$ denotes the estimated errors of combined parameters.

## 3 Robust Controller Design

This paper proposes a controller such that the errors between system output and reference model can converge to zero, and those errors are compliant with $e^{(n)} + \beta_{n-1}e^{(n-1)} + ..... + \beta_0 e = 0$. However, once the fuzzy approximate error or disturbance signal $w$ appears in the system, the model reference adaptive law will be insufficient. Therefore, a robust item is incorporated to make the following tracking performance possible for the system:

$$\int_0^T \underline{e}^T Q \underline{e} dt \le \underline{e}^T(0) P \underline{e}(0) + \frac{1}{\gamma} \tilde{\underline{\theta}}^T(0) \tilde{\underline{\theta}}(0) + \rho^2 \int_0^T w^T w dt \tag{2}$$

where P and Q are positive definite symmetric matrices; $\rho$ is a diminished coefficient; and $\gamma$ is a scale factor. Clearly, once P and Q are selected correctly in accordance with the criterion of equation (5), an asymtotically stable dissipative system will be created. Therefore, a Riccati equation can be rewritten using the Real Bound Lemma as follows:

$$PA + A^T P + Q - \left(\frac{2}{\gamma} - \frac{1}{\rho^2}\right) PBB^T P^T = 0 \tag{3}$$

When $\left(\frac{2}{\gamma} - \frac{1}{\rho^2}\right) \ge 0$ , a positive definite P matrix can be solved. Following the Hamilton-Jacobi equation, the robust control law is derived:

$$u_S(x) = -\frac{1}{\gamma} B^T P \underline{e} \tag{4}$$

## 4 Fuzzy Model Reference Adaptive Controller Design

***Step (1)***: A fuzzy basis function was initiated based on the Stone Weierstrass Theory in order to approximate the parameter of equation (3). The state variable of the system as well as membership function and number of rules were defined first. Then parameter $\underline{\hat{\theta}}$ was adjusted via the adaptive law derived from Step 6 to approach and estimate the parameters of the unknown system. If the membership function is a Gaussian function with $\bar{x}$ as mean and $\sigma$ as variance, the estimated parameter of the system can be denoted as:

$$\begin{aligned}\hat{a}_k &= \sum_{l=1}^{m} \xi_{lk}(x)\hat{\theta}_{lk} = \sum_{l=1}^{m} \hat{\theta}_{lk} \frac{\prod_{i=1}^{n} \mu_{ilk}(x_i)}{\sum_{i=1}^{m}\prod_{i=1}^{n} \mu_{ilk}(x_i)} \\ &= \sum_{l=1}^{m} \hat{\theta}_{lk} \frac{\prod_{i=1}^{n} \exp(-\frac{(x_i - \bar{x}_{ilk})^2}{2\sigma_{ilk}^2})}{\sum_{i=1}^{m}\prod_{i=1}^{n} \exp(-\frac{(x_i - \bar{x}_{ilk})^2}{2\sigma_{ilk}^2})}\end{aligned} \quad \textbf{(5)}$$

where $l = 1,....,m$ represents the number of rules; $i = 1,.....,n$ represents the number of states variables; and $k = 1,.....,n$ represents the number of parameters.

***Step (2)*****:** Properly select the values of $\beta_{n-1}, \beta_{n-2},....,\beta_0$, thereby stablizing the eigenvalues of error equation $e^n + \beta_{n-1}e^{n-1} + ..... + \beta_0 = 0$ between the system and the reference model.

***Step (3)*****:** Define the auxiliary variable $z = y_m^{(n)} - \beta_{n-1}e^{(n-1)} - ..... - \beta_0 e$; and subtract $a_n z$ from both sides of equation (3) to obtain:

$$a_n[y^{(n)} - z] = u - a_n z - a_{n-1}y^{(n-1)} - ..... - a_0 y + w \quad \textbf{(6)}$$

***Step (4)***: Let the controller output be as following in order to track the trajectory of the reference model:

$$u_e = \hat{a}_n z + \hat{a}_{n-1}y^{(n-1)} + ..... + a_0\hat{y} = \underline{v}^T \underline{\hat{a}} = \underline{v}^T \xi \underline{\hat{\theta}} \quad \textbf{(7)}$$

where $v = [z, y^{(n-1)},....., y]^T$. To increase the robustness of the system and decrease the fuzzy approximate errors and disturbance, the following robust item is added: $u_S(e) = -\dfrac{1}{\gamma} B^T P \underline{e}$.

The associated controller outputs are:

$$\begin{aligned} u &= u_e - u_s \\ a_n[y^{(n)} - z] &= a_n[y^{(n)} - y_m^{(n)} + \beta_{n-1}e^{(n-1)} ..... + \beta_0 e] \\ &= a_n[e^{(n)} + \beta_{n-1}e^{(n-1)} + ..... + \beta_0 e]\end{aligned} \quad \textbf{(8)}$$

$$\therefore \quad a_n[y^{(n)} - z] = \underline{v}^T \underline{\xi}\underline{\hat{\theta}} - \underline{v}^T \underline{\xi}\underline{\theta} + u_S + w \quad \textbf{(9)}$$

***Step (5)*****:** The tracking error equation is:

$$\begin{aligned} e^{(n)} + \beta_{n-1}e^{(n-1)} + ..... + \beta_0 e &= -\underline{v}^T \underline{\xi}\underline{\tilde{\theta}} + u_S + w \\ \underline{\dot{e}} &= A\underline{e} - B(\underline{v}^T \underline{\xi}\underline{\tilde{\theta}} - u_S - w)\end{aligned} \quad \textbf{(10)}$$

where $A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \bullet & \bullet & \bullet & \cdots & \bullet \\ 0 & 0 & 0 & \cdots & 1 \\ -\beta_0 & -\beta_1 & -\beta_2 & \cdots & -\beta_{N-1} \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

***Step (6)***: Define the Lyapunov function as:

$$V(e,\tilde{\theta}) = \frac{1}{2}\underline{e}^T P\underline{e} + \frac{1}{2\gamma}\underline{\tilde{\theta}}^T\underline{\tilde{\theta}} \qquad \textbf{(11)}$$

and then take derivation of equation (14):

$$\dot{V}(e,\tilde{\theta}) = \frac{1}{2}(\underline{\dot{e}}^T P\underline{e} + \underline{e}^T P\underline{\dot{e}} + \frac{1}{\gamma}\dot{\underline{\tilde{\theta}}}^T\underline{\tilde{\theta}} + \frac{1}{\gamma}\underline{\tilde{\theta}}^T\dot{\underline{\tilde{\theta}}})$$

$$= \frac{1}{2}(\underline{e}^T A^T P\underline{e} - \underline{\tilde{\theta}}^T\underline{\xi}^T \underline{v}B^T P\underline{e} - \frac{1}{\gamma}\underline{e}^T PBB^T P\underline{e} + w^T B^T P\underline{e} + \underline{e}^T PA\underline{e}$$

$$-\underline{e}^T PB\underline{v}^T\underline{\xi}\underline{\tilde{\theta}} - \frac{1}{\gamma}\underline{e}^T PBB^T P\underline{e} + \underline{e}^T PBw + \frac{1}{\gamma}\dot{\underline{\hat{\theta}}}^T\underline{\tilde{\theta}} + \frac{1}{\gamma}\underline{\tilde{\theta}}^T\dot{\underline{\hat{\theta}}}^T)$$

$$= \frac{1}{2}\underline{e}^T(A^T P + PA + \frac{1}{\gamma^2}PBB^T P)\underline{e} + (\frac{1}{\gamma}\underline{\tilde{\theta}}^T\dot{\underline{\hat{\theta}}} - \underline{\tilde{\theta}}^T\underline{\xi}^T\underline{v}B^T P\underline{e}) + \frac{1}{2}\underline{e}^T PBw + \frac{1}{2}w^T B^T P\underline{e}$$

$$= -\frac{1}{2}\underline{e}^T Q\underline{e} - \frac{1}{2\sigma^2}PBB^T P + \frac{1}{2}\underline{e}^T PBw + \frac{1}{2}w^T B^T P\underline{e}$$

$$= -\frac{1}{2}\underline{e}^T Q\underline{e} - \frac{1}{2}(\frac{1}{\sigma}B^T P\underline{e} - \rho w)^T(\frac{1}{\sigma}B^T P\underline{e} - \rho w) + \frac{1}{2}\rho^2 w^T w$$

$$\le -\frac{1}{2}\underline{e}^T Q\underline{e} + \frac{1}{2}\rho^2 w^T w$$

Let $(\frac{1}{\gamma}\underline{\tilde{\theta}}^T\dot{\underline{\hat{\theta}}} - \underline{\tilde{\theta}}^T\underline{\xi}^T\underline{v}B^T P\underline{e}) = 0$ to infer the adaptive law of $\hat{\theta}$ and determine $-\frac{1}{2}\underline{e}^T Q\underline{e} + \frac{1}{2}\rho^2 w^T w < 0$ based on the nature of the dissipative system. Therefore, $\dot{V}(e,\tilde{\theta}) < 0$.

***Step 7*:** The following conclusions are derived from the above inference:

**Control law:**

$$u = u_e + u_s = \underline{v}^T\xi\underline{\hat{\theta}} + \frac{1}{\gamma}B^T P\underline{e} \qquad \textbf{(12)}$$

**Adaptive law:**

$$\dot{\hat{\theta}} = \gamma\underline{\xi}^T\underline{v}B^T P\underline{e} \qquad \textbf{(13)}$$

## 5 Simulation Results

**[Example 1]:** Consider the following 2nd order nonlinear system:

$$\ddot{y} + (0.1 + 2e^{-(y^2+\dot{y}^2)})\dot{y} + \sin(y^2 + \dot{y}^2)y = 0$$

**reference model**: $\ddot{y}_m + 1.4\dot{y}_m + y_m = r$,

**system parameter:** $a_0 = \sin(y^2 + \dot{y}^2)$, $a_1 = (0.1 + 2e^{-(y^2+\dot{y}^2)})$, *and* $a_2 = 1$.

Then define variables $x_1 = y, x_2 = \dot{y}$ and use 25 rules to create the fuzzy basis function. Choosing the eigenvlues of errors equation, $\beta_0$ and $\beta_1$ are -1 and -2 located in the left plane. The auxiliary variable is defined as:

$$z = r - y_m - 1.4\dot{y}_m - 2e_2 - e_1$$

We select $Q = \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix}$, and $\gamma = 0.005, \rho = 0.01$; then solve the equation $AP + PA^T - PB(1 - \frac{1}{\gamma^2})B^T P + Q = 0$, and thus obtain $P = \begin{bmatrix} 15 & 5 \\ 5 & 5 \end{bmatrix}$.

Based on the control law and adaptive law proposed in this paper; some main simulation results are shown in Fig. 1 to Fig. 4. Here, Fig. 1 shows the response without robust control when input is chosen as $u = \underline{v}^T \xi \underline{\hat{\theta}}$ ; Fig. 2 shows the response with robust control, indicating significant improvement with respect to the tracking performance; Fig. 3 and Fig. 4 verify the estimated and approximate effects of parameters $a_0$ and $a_1$ in this architecture.

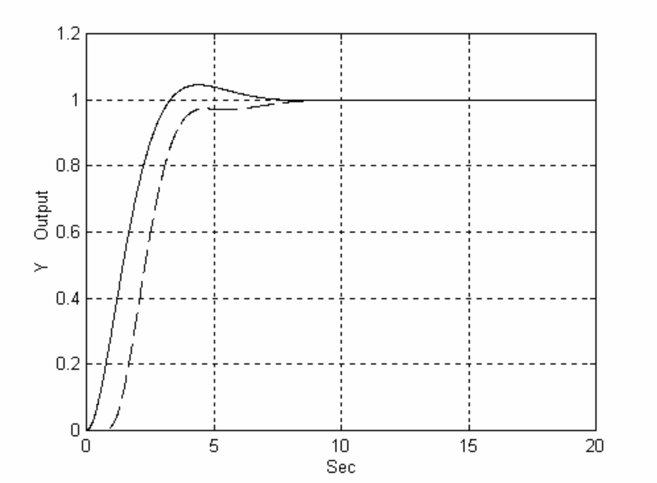


**Fig. 1.** Output response without robust controller

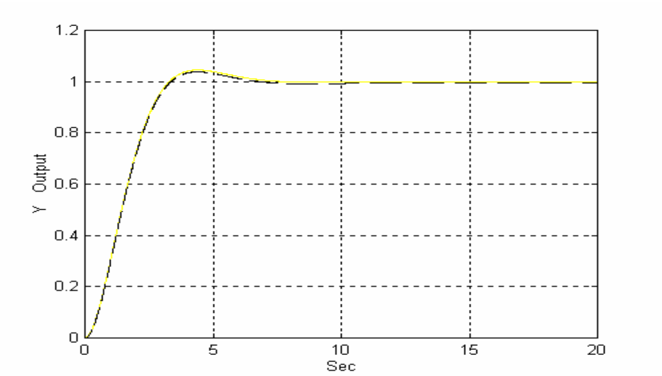


**Fig. 2.** Output response with robust controller

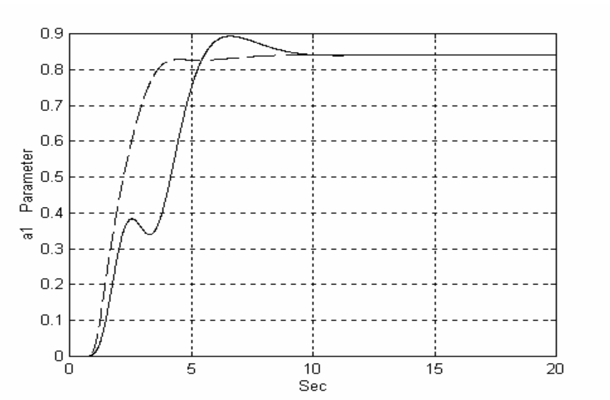


**Fig. 3.** Approximate effect of estimated parameter $a_1$

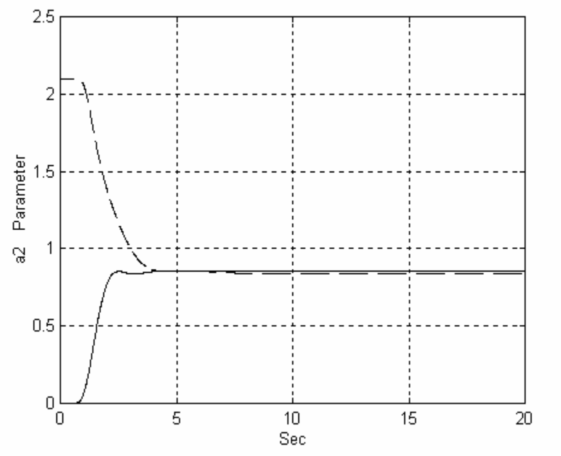


**Fig. 4.** Approximate Effect of Estimated Parameter $a_2$

Then the disturbance signal was included in the simulation process with the expected value of $w$ as 0 and the variance as 0.1. Simulation results are as follows:

Fig. 5 shows the response without robust control, indicating that the fuzzy approximate error and the disturbance diminish the tracking performance. Including robust control can overcome the effect of disturbance, and the response is shown in Fig. 6.

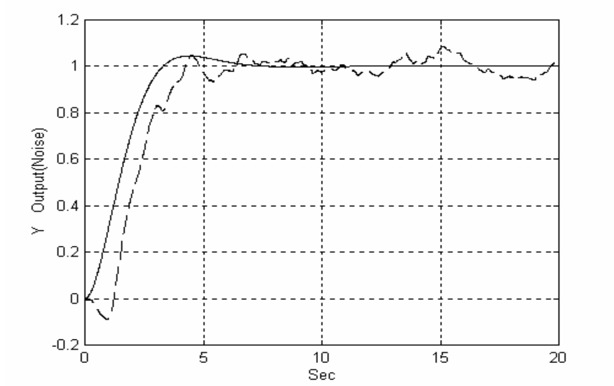


**Fig. 5.** Output response with disturbance

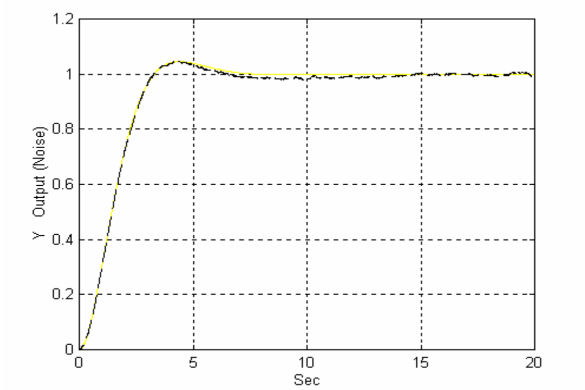


**Fig. 6.** Robust control overcome the effect of disturbance

**[Example 2]:** Consider the following 2nd order nonlinear system:

$$\ddot{y} + b\dot{y} + a\cos(y)y = u + w(t) \,,$$

which is a swaying system with disturbance and viscous friction. There, $a$ and $b$ are uncertain time-varying parameters where $a = \hat{a} + \Delta a, b = \hat{b} + \Delta b$ , $\hat{a} = 3.2$, $\Delta a = |0.3\sin(t)|$, $\hat{b} = 0.5$, and $\Delta b = |0.2\sin(t)|$ . The expected value of disturbance $w$ is 0 and the variance is 0.1; the reference model is expressed as $\ddot{y}_m + y_m = 1$. For simulation results Fig. 7 and 8 show the comparison between the response without robust control and the response with robust control.

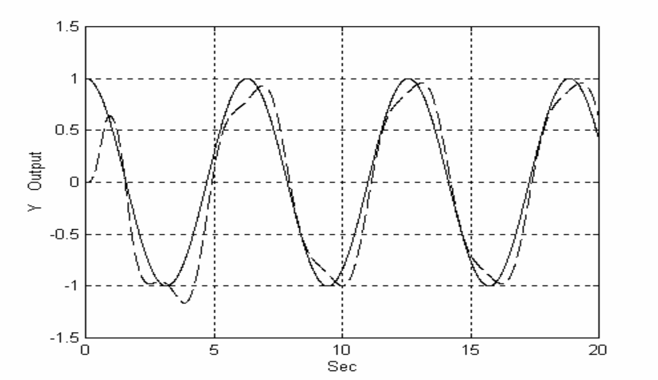


**Fig. 7.** Output response without robust control

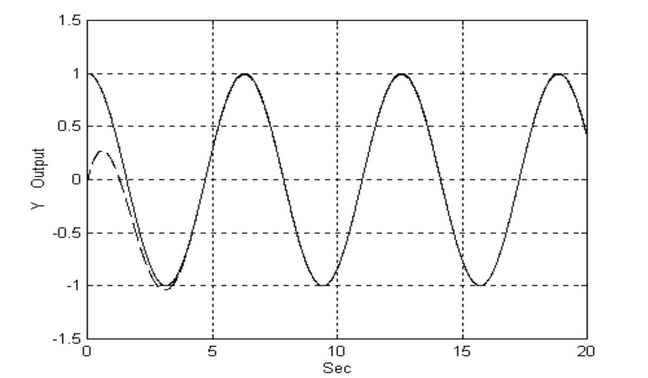


**Fig. 8.** Output response with robust control

## 6 Conclusions

The control architecture proposed in this paper is based on fuzzy basis function and has successfully incorporated the advantages of adaptive control and robust control, and thus is beneficial to the tracking control of the reference model. The contributions made by this paper include: (1) Approximating an unknown system by fuzzy basis function, thereby building up the basic of theoretical analysis for fuzzy modeling; (2) Estimating the model parameters under the structure of reference model; (3) Overcoming the drawback of fuzzy control with respect to the stability analysis by the control law and adaptive law derived from the criterion of Lyapunov stability; and (4) Eliminating the influence imposed by approximate errors and external disturbance by the robust item. In the future, researches may incorporate the powerful learning ability of the neural network to adapt the parameters of various fuzzy basis functions, thereby eliminating the approximation errors.

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