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Applying Computer Simulation to Analyze the Normal Approximation of Binomial Distribution

Horng-Jinh Chang[1] and Ming-Chen Lee[1*]

[1] Department of Management Science, Tamkang University
Tamsui 251, Taiwan, ROC
chj@fcu.edu.tw, mindy1115@yahoo.com.tw

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Abstract. Many statistical analyses were implicitly based on the normal distribution, and as a consequence, researchers would need to adopt the central limit theorem to perform the subsequent data analysis. When applying the central limit theorem, the sample size should be 30 or above in order to have the sampling distribution of sample means to be approximated to the normal distribution. Chang et al. (2006 and 2008) showed that when applying the central limit theorem, the sample size should vary depending on the probability distribution type. As a result, the present study examined if the sample size suggested by many textbooks for using the central limit theorem is appropriate. This study uses computer simulation on approximation of the binomial distribution to the normal distribution. It is to explore the minimum sample size required for a binomial distribution to approximate the normal distribution and be replaced by the normal distribution.

Keywords: binomial distribution, computer simulation, data analyzing, normal distribution

1 Introduction

The normal distribution has a symmetrical bell shape, and it can be used to describe most social, science, industrial, and research phenomena. For example, many physical, biological, social, and psychological characteristics display a normal distribution. The normal distribution is important because it makes the analysis less complicated [24] and moreover, the bell-shaped curve and the symmetrical nature of the distribution can be used for the probability model of many types of populations. According to the central limit theorem, when the sample size, $n > 30$, the sampling distribution of sample mean $\bar{x}$ will be approximated to the normal distribution. In other words, the normal distribution can be used as the approximate distribution of many types of samples under the central limit theorem [9]. In real life, there are assorted types of probability distribution, such as the unimodal vs. multimodal distributions, the symmetrical vs. asymmetrical distributions, the high vs. low skewness distributions, as well as the non-modal, non-skewed and tailless uniform distribution. While some of probability distributions have a normal distribution-like pattern, others may have a pattern that differs greatly from the normal distribution pattern.

In the social science and natural science research domain, there are random experiments that regardless of the sample size, there are only two possible outcomes, such as good vs. bad products, effective vs. non-effective drugs, head vs. tail of coin tossing, customers' like vs. dislike of a company's products, and students' presence vs. absence in a field trip. These random experiments all contain n independent and identical trials, and each trial has only two outcomes: success or failure. If the outcome probability of each trial is the same, this type of experiment is called a binomial random experiment. An excellent example of binomial random experiments from our everyday life is food safety inspection and testing, which have caught great attention because of serious food safety concerns in recent years. For this type of inspection and testing, there are only two outcomes: pass or fail, and therefore, they can be viewed as a

* Corresponding Author

type of the binomial random experiment.

How large should the sample size n be so that the central limit theorem can be applied appropriately? Most of the statistical textbooks or applied researched papers commonly assert that when the sample size is larger than 30, the sampling distribution of the sample means $\bar{x}$ may be assumed to be approximately normally distributed, even if the distribution of the population is unknown [1, 7, 19, 21-23, 25-27, 29-32]. Some statistical literature also believes that when the population distribution is continuous, unimodal and symmetric, and no matter how small the sample size is, the approximate normality can still be assumed [2-4, 28]. Nevertheless, other academic studies have found that in many realistic or in the skewed and asymmetric population cases [11-12], a sample size of 30 is not sufficient to implement the central limit theorem, and certain misleading conclusions can be produced [5-6, 8]. Therefore, it is quite significant for us to investigate the accurate sample size to support the central limit theorem. Namely, is the sample size of 30 too large or too small?

A query search in the Scopus e-database using central limit theorem and sample size as keywords returned a total of 327 related articles on the discussion of central limit theorem and sample size in recent years. However, most of the articles were discussions of the Kernel Estimator, Entropy, Convergence, Markov Chain, Monte Carlo, Covariance Matrix, Martingale, and statistics class teaching and instruction; journal articles that truly explored the central limit theorem and sample size are not that many [5-6, 10, 14]. After further studying these papers, we found that only two articles related to the probability distribution. They especially explored the application of the central limit theorem on the sample size of Weibull and Gamma distributions, respectively [5-6]. In the present study, therefore, the investigators examined the appropriateness of using the sample size suggested by general textbooks for determining whether the central limit theorem can be used or not. It was done using the modern computer technology to test the minimum simple size and properties required for the means of the binomial distribution to be approximated to the normal distribution. The objective of the study was to explore, at 5%, 10%, and 20% error levels, the minimum sample size n required for the normal approximation to the binomial distribution and to have the binomial distribution replaced by the normal distribution. Figures were also produced to provide other fields as reference material when applying the central limit theorem.

2 Statistical Tests and Simulation Steps

Under the central limit theorem, if one assumes that the sample mean $\overline{X_n}$ is the mean of a random sample $X_1, X_2, ..., X_n$ of size n of a population that has a mean of u and a variance of σ^2 (both greater than 0), then when n is close to infinite positive, the random distribution of $\overline{X_n}$ will become approximated to either a normal distribution where the mean is u and the standard deviation is $\sigma\sqrt{n}$ or to the distribution of the standard normal $N(0,1)$.

$$Z = \frac{\overline{X}_n - u}{\sigma/\sqrt{n}} = \frac{\sum_{i=1}^{n} X_i - nu}{\sqrt{n}\sigma} \to N(0,1\ )\ . \tag{2.1}$$

Assume X as a discrete random variable with a probability mass function:

$$f(x) = C_x^n p^x q^{n-x}\ ,\ x > 0,\ 0 < p < 1. \tag{2.2}$$

In this case, it can be referred to as a binomial distribution with two parameters n and p [20], denoted by $X \sim B(n,p)$. In Equation $C_x^n = n!/x!(m-x)!$, where n is the number of trials, x is the number of success , p is the probability of success, and q is the probability of failure (=1−p). If parameters n and p of the binomial probability distribution are known, then the probability of each variable in the binomial probability distribution can be obtained.

The Shapiro–Wilk W-test proposed by Shapiro and Wilk [16] was used for normality testing, and the definition of the W-test is presented below:

$$W = \left\{ \sum_{i=1}^{h} a_{in} \left(x_{n-i+1} - x_{(i)} \right) \right\}^2 / \sum_{i=1}^{n} \left(x_i - \bar{x} \right)^2, x_{(1)} \leq .... \leq x_{(n)}. \quad \textbf{(2.3)}$$

When n is an even number, $h=n/2$, and if n is an odd number, $h=(n-1)/2$. Shapiro and Wilk also provided a cross-reference table for parameters α_{in} . Compared with other normality tests, the Shapiro–Wilk W-test is more sensitive. That is, it is applicable even for a small sample size ($n < 20$) or if there are outliers [17, 18]. Through computer implementation, Royston [15] extended the Shapiro–Wilk W statistic from small sample to large sample applications, Pearson et al. [13] also mentioned that compared with other normality testing methods, the W-test not only remains highly sensitive but also has the highest normality test power even when biased. Therefore, the study used the W-test statistics for the normality testing of the random distribution of the sample mean.

The study used the built-in binomial distribution random function of the Excel statistic program to carry out the random sampling simulation. With parameter $p \in \{0.05, 0.1, 0.2, 0.3, \ldots\ldots, 0.9, 0.95\}$ and a sample size of , $n \in \{2, 3, \ldots\ldots, 300\}$, a random number function was used to randomly sample 200 sample sets, which provided a set of sample means $\bar{X}_{n,1}, \bar{X}_{n,2}, \ldots\ldots, \bar{X}_{n,200}$. Next, the Shapiro–Wilk W-test was employed to test if the 200 sample means meet the normality condition (at a significance level of $\alpha=0.05$). A test result, i.e., accepting or rejecting the normality assumption, was generated for each sample size n, and if the normality assumption is rejected, it would be viewed as successful. If the above test is performed for 300 times for each sample size n, then 300 Bernoulli trial results will be generated, which can be viewed as a new binomial sample set of a sample size of 300. In the study, the test result m (number of rejection) was used as a normality indicator, and the data were plotted to explore the relationship as well as the evaluation function. A total of 29,798,340,000 random numbers [=200×(2+3+4+……+300)×11×300] were generated in this study, and the normality test was performed repeatedly for 986,700 times (=299×11×300).

3 Simulation Results

Using the above simulation method and statistical test, the investigators did a computer simulation using $p \in \{0.05, 0.1, 0.2,\ldots,0.9, 0.95\}$ and $n = 2, 3, 4,\ldots 300$ The results are presented in Table 1 at the Appendix. It can be found from Table 1 that under the binomial distribution and with $p = 0.05$, the normality assumption cannot be accepted when the sample size was smaller than 162. Even with a sample size of 300, there were still 187 out of 300 times when the normality assumption was rejected. For $p = 0.1$, the normality assumption cannot be accepted when the sample size was smaller than 86, but when the sample size was 300, the number of times rejecting the normality assumption was reduced to 74. For $p = 0.2$, the normality assumption cannot be accepted when the sample size was smaller than 43. For $p = 0.3$, the normality assumption cannot be accepted when the sample size was smaller than 30. For $p = 0.5$, the number of normality assumption rejection was reduced to 11 (out of the 300 tests) when the sample size was smaller than 25. For $p = 0.6$, the normality assumption cannot be accepted when the sample size was smaller than 25, and out of the 300 tests, the number of times rejecting the normality assumption began to increase. For $p = 0.8$, the normality assumption cannot be accepted when the sample size was smaller than 44. For $p = 0.95$, the normality assumption cannot be accepted when the sample size was smaller than 77. For $p = 0.95$, the normality assumption cannot be accepted when the sample size was smaller than 167, and out of the 300 tests, the number of times that the normality assumption was rejected was increased to 187. Therefore, when p got further away from 0.5, the sample size had to be substantially increased to meet the normality requirement for employing the central limit theorem.

The investigators also observed the simulation outcomes when the sample size was fixed to 30. It was found that when p was 0.05, 0.1, 0.2, and 0.3, the normality assumption was rejected 300 out of the 300 tests. When $p = 0.4$, the number of rejection m was dropped to 288; when $p = 0.5$, the number of rejection m was dropped to 286; when $p = 0.6$, the number of rejection m was increased to 294; when $p = 0.7$, the number of rejection m was further increased to 298; and when $p = 0.8$, the number of rejection m was back to 300. It can be found from the 300 test outcomes that even if $p = 0.5$, the number of rejection would start to drop from 300 when the sample size was greater than 25.

At this stage, according to the simulation outcomes presented above, a greater sample size would imply that the sampling distribution of sample means would be more approximated to the normal distribution. The closer the binomial distribution parameter p is to 0.5 , the higher the normal approximation speed is. Therefore, the single condition, i.e., with a sample size greater than 30 for using the central limit theorem, is apparently not appropriate for the binomial distribution.

4 Discussions

In Fig. 1, the W-test outcomes of different sample size n of different parameters $p \in \{0.05, 0.1, 0.2, 0.3, \ldots, 0.9, 0.95\}$ under the binomial distribution were plotted into a line graph. The x-axis represented the sample size n, while the y-axis represented the number of normality assumption rejection m. It can be found from Fig. 1 that all the curves had a negative slope, meaning that a larger the sample size was associated with better normal approximation to the random sampling distribution of sample means. When treating p=0.5 as the center and with the two ends p = 0.05 and p = 0.95, and the number of times rejecting the normality assumption m would start to decrease only when the sample size was greater than 160. For p = 0.1 and p = 0.9, m would begin to decrease when the sample size was greater than 85. Moreover, when the binomial distribution parameter p got closer to 0.5 (e.g., when p = 0.3, 0.4, 0.6, 0.7), the curve would become closer to the two axes, and when p was 0.5, the curve was much closer to the two axes, meaning the highest speed of normal approximation.

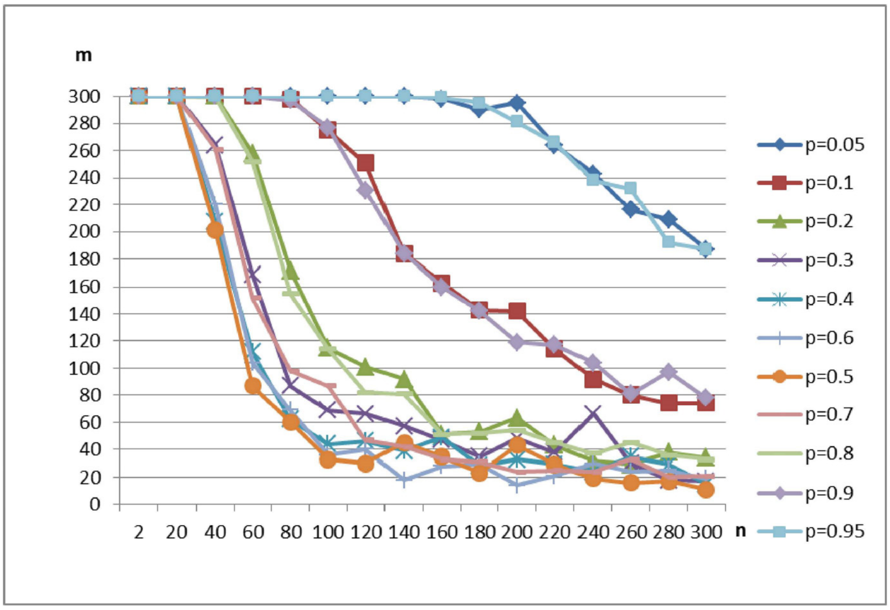


Fig. 1. The relationship between n, the sample size, and m, the number of times rejecting the normality assumption under the binomial distribution.

Meanwhile, it can be found from Fig. 1 that the number of times rejecting the normality assumption m only started to drop from 300 when the sample size n was increased to a certain level. In other words, with the various values of the binomial distribution parameter p, there was a tendency between m, the number of times of rejection, and n, the sample size. The study used a regression model to display the tendency. It was the inverse regression model ($m = b_0 + b_1 / n + \varepsilon$) that was used to find out the approximate curve. Make $\hat{m}$ the number of rejection to find out the estimated value using the approximate curve of the inverse regression model. It was found that the sample size and the number of rejection had the following relationship: ($\hat{m} = b_0 + b_1 / n$) and $n \geq 2$. Make k the number of times repeating the W-test for normality (k = 300 in the study), and $\overline{m} = \min\{\hat{m}, k\}$ was used to estimate the number of rejection, m. The estimated regression coefficients were shown in Table 2, and the approximate regression curve was plotted used the values (See Fig. 2).

Table 2. Inverse regression model of the binomial distribution

P	R^2	F	P	b_0	b_1
0.05	0.896	1181.355	0.000	56.654	42495.315
0.1	0.935	3077.868	0.000	-29.932	30839.066
0.2	0.966	7324.573	0.000	-27.837	15665.724
0.3	0.951	5218.513	0.000	-20.399	10508.851
0.4	0.948	4987.122	0.000	-18.829	8637.910
0.5	0.943	4545.461	0.000	-19.172	8189.774
0.6	0.945	4703.603	0.000	-18.860	8556.751
0.7	0.948	4891.549	0.000	-23.928	10738.014
0.8	0.972	8966.035	0.000	-24.715	15329.531
0.9	0.952	4431.786	0.000	-15.584	28424.354
0.95	0.897	1152.809	0.000	46.952	45125.497

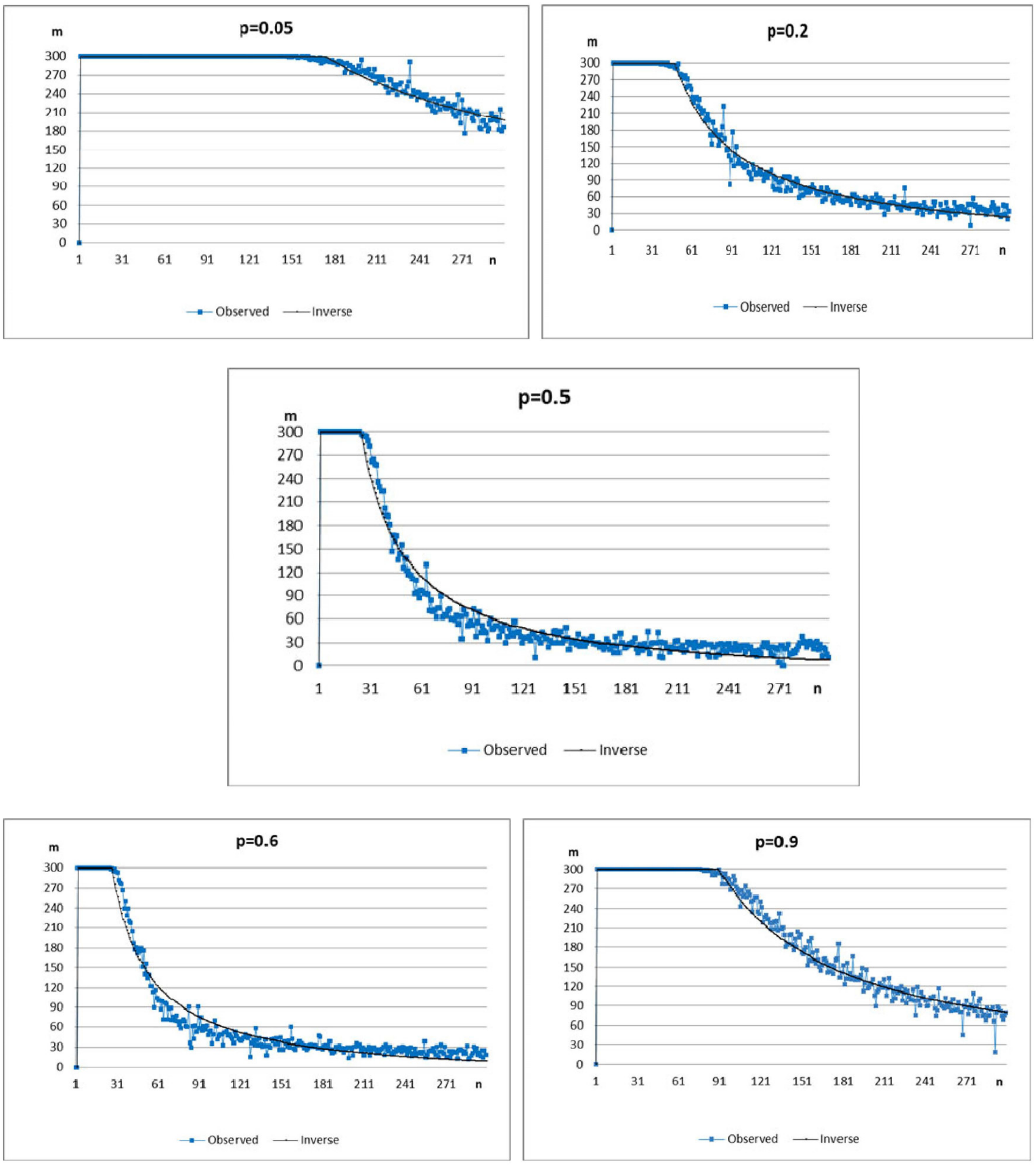
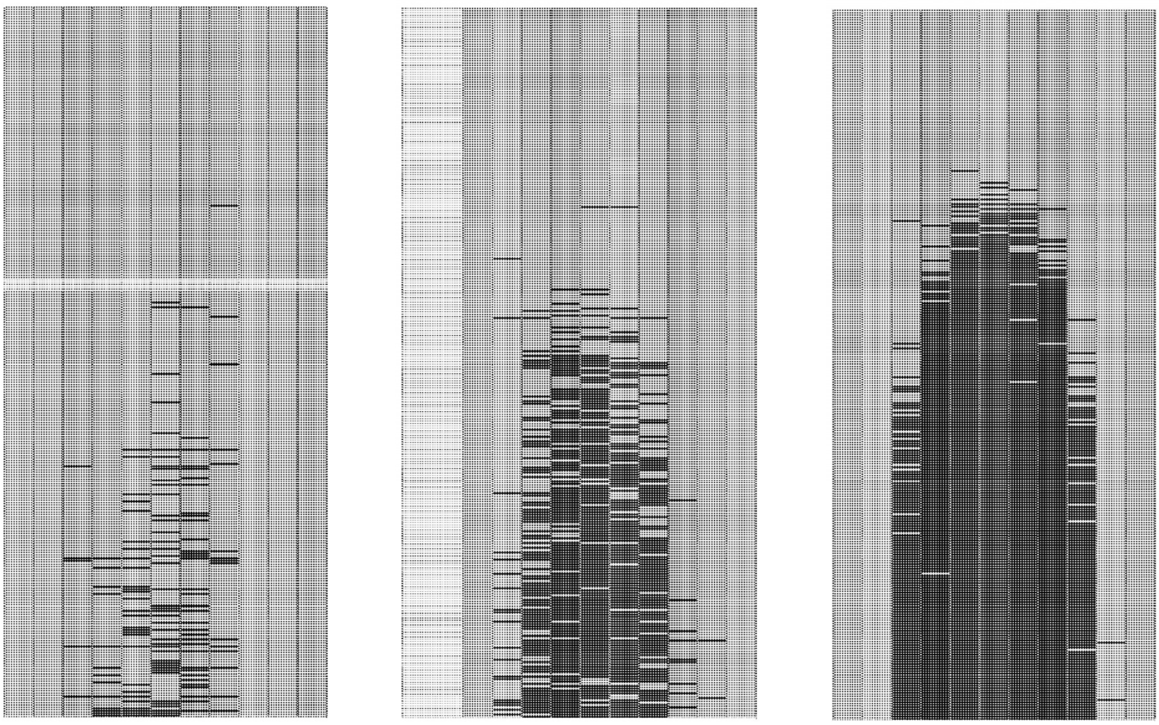


Fig. 2. Curve of the inverse regression model of the binomial distribution

It can be found from Table 2 that for each parameter p in the binomial distribution, there was a significant association between the number of rejection and the sample size. When p = 0.05, the relation can be expressed as $\hat{m}$ = 56.654+42495315/n and. R^2 = 0.896. Thereafter, the speed of normal approximation accelerated, but the fast increase of R^2 did not slow down until p = 0.2. When p = 0.5, the lowest speed was reached (R^2 = 0.943), and then R^2 increased gradually again until p = 0.8, resulting a long and flat tails of the regression curve (See Fig. 2). Overall, the degree of fit of the regression model was optimal and good.

It is mentioned in Chapter 2 that m was the number of times that the normality assumption was rejected, i.e., the number of success out of the 300 times of Bernoulli trial. In Fig. 3, each small square denotes the result of the 300 times of Bernoulli trial, and the shadow part denotes that the number of rejection of the normality assumption out of the 300 times of Bernoulli trial was greater than the required rejection rate m' set by the investigators of the study (the required rejection rate can be viewed as an equivalence of the significance level of normality testing). As for the black part, it denotes that the number of rejection was smaller than the required rejection rate of the study. Take Fig. 3(a) as an example, the required rejection rate m' was set to be 5 (i.e., $m \le 15$), and in Fig. 3(b), the rejection rate m' was 10 (i.e., $m \le 30$). In Fig. 3(c), the rejection rate m' was 20 (i.e., $m \le 60$).



(a) m'=5 (b) m'=10 (c) m'=20

Fig. 3. Bernoulli trial results of different required rejection rate m' under the binomial distribution, where the x axis was parameter p, and the y axis was the sample size n.

After altering the required rejection rate, a significant trend appeared, as shown in Fig. 3, and the trend was especially apparent in Fig. 3(c). The histogram displayed a normal distribution with p = 0.5 as the axis of symmetry. As a result, the investigators became interested in capturing the relation between the parameter p and the sample size n under the binomial distribution. It can be found from the computer simulation result that when the binomial distribution parameter p was close to 0.5, the speed of normal approximation increased. In other words, the sample size would decrease when p gets closer to 0.5 and would increase when p gets further away from 0.5.Therefore, the investigators used the linear regression model ($n = b_0 + b_1 \times |p - 0.5|$) to find out the approximate regression curve of parameter p and sample size n. It can be found in Fig. 3(a) that because the number of rejection smaller than the required rejection rate (the black part) set by the investigators and decreased, the data were discrete, and data from the first rejection was used to find out the approximate curve. It can be found in Fig. 3(b) and 3(c) that because the number of rejection smaller than the required rejection rate (the black part) set by the investigators and increased, the data were continuous, and all data from rejection were used to find out the approximate curve. As shown in Table 3 and Fig. 4 that when $m'=5$, the relation can be expressed as $n=119.892+24.942\times|p-0.5|$. When $m'=10$, the relation can be expressed as $n=76.695+34.522\times|p-0.5|$. When $m'=10$ can be expressed as $n=35.032+23.930\times|p-0.5|$.

Table 3. The regression model of different required rejection rates in binomial distribution

m'	R^2	F	P	b_0	b_1
5	0.769	26.585	0.001	119.892	24.942
10	0.801	1194.229	0.000	76.695	34.552
20	0.858	1791.953	0.000	35.032	23.930

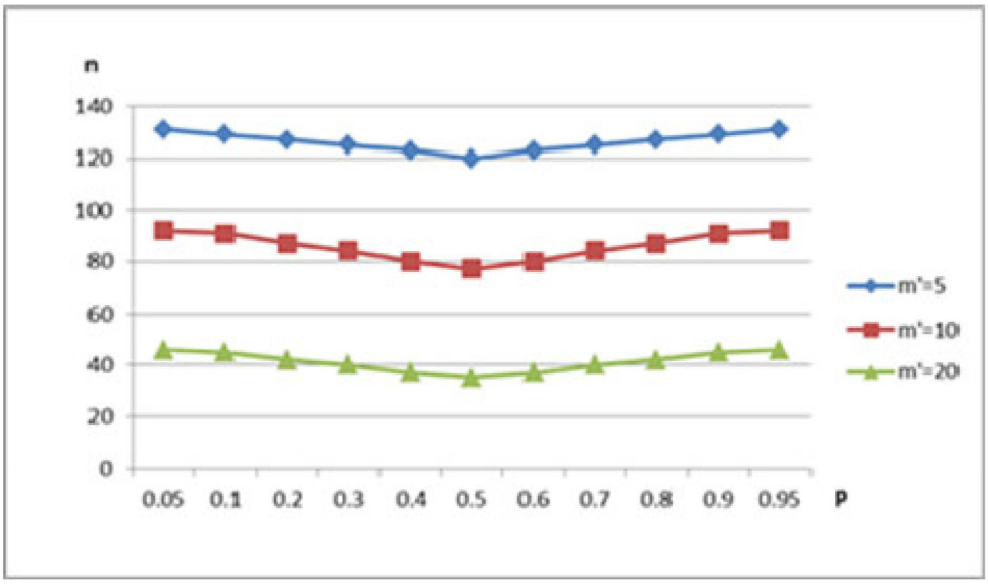


Fig. 4. The regression curve of binomial distribution parameter p and sample size n

According to the regression model presented in Table 3, the value of each p in the binomial distribution can be estimated, and when the required rejection rate m' varied, the minimum sample size required for adopting the central limit theorem can be obtained. See Table 4 for the result. It can be found in Table 4 that a binomial distribution parameter p closer to 0.5 indicates a better eligibility for using the central limit theory because the required sample size for the normal distribution decreases as the parameter p gets closer to 0.5. Take $m'\leq 5$ as an example, when p = 0.5, the minimum sample size required for using the central limited theorem was 120. When $m'\leq 10$ and p = 0.5, the minimum sample size required for using the central limited theorem was reduced to 77. When $m'=20$ and p = 0.5, the minimum sample size required for using the central limit theorem was further reduced to 35. Nevertheless, the general condition, i.e., whether the sample size n is greater than 30, was not applicable when $m'=5$ (equivalent to a significance level of 0.05 in the normal distribution) and p = 0.5, when $m'=10$ and p = 0.5, or when $m'=20$ and p = 0.5.

Table 4. Minimum sample size *n* required for using the central limit theorem in the binomial distribution

p \ n \ m'	5	10	20
0.05	131	92	46
0.1	129	91	45
0.2	127	87	42
0.3	125	84	40
0.4	123	80	37
0.5	120	77	35
0.6	123	80	37
0.7	125	84	40
0.8	127	87	42
0.9	129	91	45
0.95	131	92	46

5 Conclusions

Many statistical analyses were implicitly based on the normal distribution, and as a result, the subsequent data analysis often needs to rely on the central limit theorem. The general criterion for using the central limited theorem is to have a sample size greater than 30, so the random sampling distribution of sample means would be approximated to the normal distribution. It can be found in Table 4 that this sample size requirement was too small for the binomial distribution. For the approximate rate $(1-m')$ to reach 80%, the required sample size should be increased as p gets further away from 0.5 (i.e., a larger $|p-0.5|$). When $p = 0.5$, a sample size of 35 would be required, and when p was increased to $p = 0.05$ or $p = 0.95$, a sample size of 46 would be required. To obtain an approximate rate close to 90%, the required sample size has to be increased to 77 when $p = 0.5$ and 92 when $p = 0.05$ or $p = 0.95$. If the approximate rate is nearly 95%, then the required sample size for $p = 0.5$ would be 120, and then increased to 131 for $p = 0.05$ or $p = 0.95$. Therefore, for $m'=5$ (equivalent to a significance level of 0.05 in the normal distribution), $m'=10$, or $m'=20$, no p can have the condition, i.e., a sample size n greater than 30, satisfied.

In addition, there is also the De Moivre Laplace theorem: When $n \to \infty$, the binomial distribution will be approximated to the normal distribution. To apply the theory, the sample size n has to satisfy $np > 5$ and $n(1-p) > 5$. In other words, when $p=0.5$, $np > 5$, and $n(1-p) > 5$, n only needs to be greater than 10. Nonetheless, it was shown in the simulation presented in Table 4 that a sample size of 10 was not larger enough. When $p=0.05$ or $p=0.95$, n has to be greater than 100 in order to satisfy the conditions of $np > 5$, and $n(1-p) > 5$. An n greater than 100 can satisfy the approximate rate of 0.90 and 0.80, as shown in Table 4, but if the approximate rate is 0.95, a sample size of 100 would still be too small.

When applying the central limit theorem, most general statistics textbooks, applied papers, and researchers use the “whether sample size is greater than 30” criterion to assume the sampling distribution of the sample mean to be approximately a normally distributed, and thereby replaced by a normal distribution. Yet studies published by Chang et al. on the Weibull distribution in 2006 and the Gamma distribution in 2008, plus the present paper’s binomial distribution all indicate that, when using computer simulations to explore normal distribution approximations, the sample size required for the approximate normal distributions should vary depending on the distribution type. That is, when the central limit theorem is applied for the different probability distributions, the minimum required and reasonable sample size should also be different.

Currently, commonly-used statistical software such as MINITAB, STATISTICA, JMP, SPSS, SAS, and EXCEL (which was used in the paper), all provide a variety of common statistical analysis, and their operation is easy. However, statistical software packages are limited in their number of individual data output, so it is not convenient or feasible to perform larger-scale simulations. If follow-up studies can use R, PYTHON, JULIA, JAVA, or other more functional software or programs, not only will the scale and efficiency of the simulations be increased, but they can also be used to compare with this study to see whether the results agree.

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Appendix

Table 1. W test results of Binomial distribution as p and n varies.(p =0.05, 0.1, 0.2, …, 0.9, 0.95, n =2, 3, 4, …, 300; Numbers in the table are reject frequency of repeating 300 W tests)

Sample Size (n)	p										
	0.05	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	0.95
2	300	300	300	300	300	300	300	300	300	300	300
3	300	300	300	300	300	300	300	300	300	300	300
4	300	300	300	300	300	300	300	300	300	300	300
5	300	300	300	300	300	300	300	300	300	300	300
6	300	300	300	300	300	300	300	300	300	300	300
7	300	300	300	300	300	300	300	300	300	300	300
8	300	300	300	300	300	300	300	300	300	300	300
9	300	300	300	300	300	300	300	300	300	300	300
10	300	300	300	300	300	300	300	300	300	300	300
11	300	300	300	300	300	300	300	300	300	300	300
12	300	300	300	300	300	300	300	300	300	300	300
13	300	300	300	300	300	300	300	300	300	300	300
14	300	300	300	300	300	300	300	300	300	300	300
15	300	300	300	300	300	300	300	300	300	300	300
16	300	300	300	300	300	300	300	300	300	300	300
17	300	300	300	300	300	300	300	300	300	300	300
18	300	300	300	300	300	300	300	300	300	300	300
19	300	300	300	300	300	300	300	300	300	300	300
20	300	300	300	300	300	300	300	300	300	300	300
21	300	300	300	300	300	300	300	300	300	300	300
22	300	300	300	300	300	300	300	300	300	300	300
23	300	300	300	300	300	299	300	300	300	300	300
24	300	300	300	300	300	300	300	300	300	300	300
25	300	300	300	300	300	300	300	300	300	300	300
26	300	300	300	300	**298**	**297**	**298**	300	300	300	300
27	300	300	300	300	297	295	299	300	300	300	300
28	300	300	300	300	299	295	295	300	300	300	300
29	300	300	300	300	296	294	298	300	300	300	300
30	**300**	**300**	**300**	**300**	**288**	**286**	**294**	**298**	**300**	**300**	**300**
31	300	300	300	**299**	281	281	293	299	300	300	300
32	300	300	300	299	276	262	282	300	300	300	300
33	300	300	300	297	273	264	278	**296**	300	300	300
34	300	300	300	298	275	259	276	295	300	300	300
35	300	300	300	292	270	258	267	296	300	300	300
36	300	300	300	288	259	236	239	291	300	300	300
37	300	300	300	291	227	230	250	290	300	300	300
38	300	300	299	283	239	224	229	280	300	300	300
39	300	300	300	280	227	224	239	275	300	300	300
40	300	300	300	264	207	201	220	261	299	300	300
41	300	300	299	270	223	193	218	271	300	300	300
42	300	300	297	262	203	192	204	268	300	300	300
43	300	300	300	263	209	181	187	254	297	300	300
44	300	300	**296**	250	193	147	178	258	300	300	300
45	300	300	295	240	198	168	181	229	**299**	300	300
46	300	300	295	235	166	159	172	235	295	300	300
47	300	300	295	229	176	167	175	222	295	300	300
48	300	300	292	211	168	136	180	222	293	300	300
49	300	300	296	211	183	144	151	208	287	300	300
50	300	300	290	201	165	155	176	213	289	300	300

Table 1. (Continued 1)

Sample Size (n)	p										
	0.05	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	0.95
51	300	300	298	202	151	126	141	208	282	300	300
52	300	300	280	169	139	139	156	193	241	300	300
53	300	300	278	196	144	123	135	197	276	300	300
54	300	300	276	202	136	118	139	191	269	300	300
55	300	300	273	160	134	116	123	178	277	300	300
56	300	300	277	165	132	112	123	178	266	300	300
57	300	300	257	172	127	92	113	175	257	300	300
58	300	300	272	160	118	110	90	160	248	300	300
59	300	300	258	158	114	95	116	157	239	300	300
60	300	300	258	169	112	87	104	152	252	300	300
61	300	300	253	96	88	97	100	176	232	300	300
62	300	300	238	152	103	96	101	151	245	300	300
63	300	300	229	139	102	92	87	92	237	300	300
64	300	300	236	122	112	130	99	125	225	300	300
65	300	300	238	133	92	91	72	139	236	300	300
66	300	300	222	102	80	70	96	117	214	300	300
67	300	300	235	118	101	84	94	148	228	300	300
68	300	300	218	134	94	69	72	110	206	300	300
69	300	300	211	121	65	71	88	122	208	300	300
70	300	299	214	122	96	62	89	120	194	300	300
71	300	299	197	107	75	73	71	117	219	300	300
72	300	300	201	113	81	73	75	104	206	300	300
73	300	300	208	121	82	89	70	104	200	300	300
74	300	300	199	111	70	62	77	95	192	300	300
75	300	300	171	100	80	65	69	120	197	300	300
76	300	300	155	98	81	64	63	108	176	300	300
77	300	300	195	98	72	70	58	72	173	300	300
78	300	300	172	109	76	72	69	103	167	**299**	300
79	300	298	179	102	74	60	72	91	179	299	300
80	300	298	172	87	62	60	69	98	155	297	300
81	300	298	153	83	52	59	61	95	174	297	300
82	300	299	161	87	60	64	60	94	148	299	300
83	300	298	171	97	59	54	91	82	178	298	300
84	300	300	187	73	44	63	37	90	161	297	300
85	300	300	222	177	151	35	30	15	197	298	300
86	300	300	165	116	56	71	61	101	173	291	300
87	300	**297**	145	85	62	65	42	79	145	295	300
88	300	292	146	93	63	51	62	94	144	291	300
89	300	293	134	90	56	54	53	75	143	294	300
90	300	297	82	86	110	57	91	70	160	294	300
91	300	295	126	72	52	53	58	81	150	299	300
92	300	296	177	39	46	72	74	90	128	294	300
93	300	287	115	84	58	38	55	90	132	277	300
94	300	288	135	76	45	56	61	83	126	288	300
95	300	280	150	95	61	68	56	77	117	293	300
96	300	281	120	69	64	43	62	70	130	277	300
97	300	284	127	73	56	52	48	72	115	286	300
98	300	283	119	63	46	45	53	66	128	284	300
99	300	274	120	72	57	43	55	43	116	268	300
100	300	275	115	69	44	33	36	87	114	277	300
101	300	282	113	61	52	55	49	56	121	289	300
102	300	276	117	66	55	60	70	68	136	283	300
103	300	281	116	70	36	47	61	57	134	273	300
104	300	274	104	64	54	48	41	78	100	269	300
105	300	275	101	69	60	52	46	63	123	263	300

Table 1. (Continued 2)

Sample Size (n)	p										
	0.05	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	0.95
106	300	264	92	74	53	51	49	70	112	243	300
107	300	267	118	54	56	38	49	55	106	267	300
108	300	267	109	65	56	48	33	66	103	258	300
109	300	268	100	72	54	45	52	50	96	260	300
110	300	260	102	64	56	53	52	72	115	274	300
111	300	264	101	67	43	29	54	53	97	257	300
112	300	250	108	54	48	38	47	45	92	265	300
113	300	248	107	58	59	48	49	56	104	260	300
114	300	245	99	76	33	42	45	61	111	251	300
115	300	245	89	60	47	39	42	58	84	235	300
116	300	244	97	63	55	57	37	43	103	253	300
117	300	237	101	56	45	43	53	50	85	257	300
118	300	248	94	56	46	44	46	59	106	257	300
119	300	250	104	60	43	38	42	54	102	235	300
120	300	251	101	66	46	30	40	47	82	231	300
121	300	243	107	39	38	37	39	47	101	251	300
122	300	236	78	68	35	37	41	49	94	242	300
123	300	233	97	48	40	36	44	52	94	218	300
124	300	235	73	61	34	38	49	49	89	226	300
125	300	231	91	54	55	33	44	47	86	230	300
126	300	213	73	53	34	41	43	48	101	223	300
127	300	224	86	55	47	42	46	47	105	223	300
128	300	229	72	46	38	11	16	48	89	217	300
129	300	221	87	55	30	37	37	44	70	208	300
130	300	226	92	46	49	35	40	41	76	217	300
131	300	227	95	51	37	30	34	54	92	219	300
132	300	207	70	30	49	43	58	24	68	208	299
133	300	225	76	53	37	40	40	51	92	220	300
134	300	205	96	57	40	35	33	49	93	206	300
135	300	206	94	46	37	35	43	39	92	232	300
136	300	208	71	55	35	24	34	44	76	208	300
137	300	217	86	43	41	33	31	47	87	210	300
138	300	182	84	36	35	30	32	51	86	199	300
139	300	209	79	46	42	44	41	43	81	180	300
140	300	184	92	57	39	45	18	42	81	184	300
141	300	196	88	50	45	29	32	38	70	185	300
142	300	205	58	48	37	41	30	59	65	199	300
143	300	191	75	46	33	44	41	49	72	200	300
144	300	198	62	39	36	30	38	49	89	184	300
145	300	196	64	44	35	34	42	55	69	175	300
146	300	187	76	46	28	49	35	48	60	193	299
147	300	179	73	53	32	36	43	49	68	180	300
148	300	171	68	40	21	21	35	42	66	204	300
149	299	188	68	45	46	29	26	56	85	195	299
150	300	180	76	44	29	31	44	40	66	200	300
151	299	182	70	52	21	33	27	39	74	171	299
152	300	178	81	36	27	27	26	10	79	169	299
153	300	187	76	32	24	41	38	28	73	175	299
154	300	184	74	45	30	25	36	39	78	179	300
155	298	161	68	44	33	30	31	33	69	153	300
156	299	164	63	47	21	35	39	26	63	189	299
157	299	160	64	43	37	25	30	48	65	160	300
158	300	190	69	49	32	29	60	40	50	194	298
159	299	157	75	44	32	33	42	55	65	171	298
160	298	163	51	47	49	35	28	33	51	160	299

Table 1. (Continued 3)

Sample Size (n)	p										
	0.05	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	0.95
161	300	169	68	43	43	38	26	50	60	156	298
162	300	167	56	40	27	28	34	48	67	174	298
163	**296**	162	75	38	27	30	35	42	65	161	299
164	295	167	72	33	26	27	38	29	56	151	300
165	298	173	65	25	30	30	36	39	62	145	294
166	298	167	67	36	29	24	33	41	61	153	300
167	296	157	52	31	35	27	29	41	67	162	300
168	295	152	49	30	31	21	32	29	64	163	**294**
169	297	161	60	28	31	24	26	37	51	158	296
170	294	165	68	35	34	35	29	43	61	142	297
171	298	166	55	43	21	31	33	47	50	154	299
172	289	146	62	48	10	21	34	51	56	143	298
173	292	158	52	44	28	26	31	49	54	153	291
174	297	146	52	30	21	17	31	37	69	140	288
175	295	139	50	29	38	38	33	32	50	151	290
176	293	146	53	40	28	25	32	33	66	161	297
177	293	145	53	45	33	17	29	38	52	163	290
178	291	146	57	47	59	42	47	31	49	185	290
179	294	150	55	25	25	30	45	46	45	134	295
180	290	143	53	35	28	23	29	31	52	142	295
181	290	149	46	44	35	24	27	43	59	153	290
182	290	142	64	34	27	26	32	35	44	123	290
183	287	132	64	35	26	28	33	41	63	132	290
184	293	128	63	35	22	27	31	23	55	157	289
185	292	123	51	30	32	31	33	44	48	132	283
186	289	153	58	34	30	36	38	39	50	141	288
187	288	134	55	37	35	22	26	28	46	131	287
188	275	146	45	78	34	17	21	31	32	165	299
189	279	121	52	43	27	27	23	38	51	135	289
190	289	131	52	42	28	24	31	36	69	130	288
191	285	135	48	29	21	20	29	37	60	134	276
192	286	138	58	38	35	28	30	25	50	130	287
193	275	120	55	36	27	23	32	28	61	132	286
194	283	120	40	43	32	44	26	16	31	138	283
195	278	128	42	26	22	16	29	25	58	149	279
196	279	105	51	30	28	28	22	21	50	113	276
197	276	129	57	25	33	30	28	38	48	125	284
198	285	118	53	37	33	29	27	34	42	146	281
199	276	128	50	24	26	30	30	36	51	117	276
200	295	142	63	48	33	43	14	24	54	119	281
201	278	120	54	29	25	12	30	34	59	125	276
202	278	120	48	40	30	31	28	34	49	123	281
203	274	121	57	36	32	11	18	25	54	131	271
204	278	125	42	39	19	30	31	25	50	106	279
205	280	102	52	52	27	24	31	28	41	89	282
206	270	115	29	43	32	24	36	24	45	112	281
207	271	118	42	30	22	18	32	25	47	115	268
208	268	108	45	34	29	24	34	30	51	125	277
209	280	107	38	42	25	22	25	28	31	121	267
210	256	111	47	29	19	32	30	28	51	118	274
211	268	120	48	27	29	33	18	25	50	131	277
212	262	112	49	31	28	22	27	36	52	119	266
213	268	131	48	25	24	21	26	31	48	106	264
214	267	126	60	30	22	26	31	38	48	117	260
215	268	118	46	26	28	18	18	29	49	133	268

Table 1. (Continued 4)

Sample Size (n)	p										
	0.05	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	0.95
216	263	113	41	29	19	20	25	33	45	119	256
217	251	111	44	31	24	31	30	34	54	97	265
218	250	110	41	25	26	27	30	34	44	113	251
219	242	115	37	31	32	30	27	43	53	107	264
220	264	114	43	38	29	29	21	25	45	117	266
221	263	115	48	42	32	23	27	31	38	110	265
222	246	96	76	22	21	31	21	47	49	113	261
223	255	102	41	39	33	13	26	39	33	120	265
224	248	113	43	25	31	25	28	33	56	98	268
225	239	96	39	23	27	25	31	24	43	110	264
226	254	107	44	37	32	30	19	39	33	115	250
227	255	99	46	31	20	29	34	29	58	94	259
228	259	110	43	32	18	27	27	25	42	113	247
229	243	122	46	28	20	15	24	31	55	103	241
230	244	102	46	35	24	12	31	33	40	99	262
231	241	121	31	37	30	27	21	12	41	99	254
232	251	89	39	30	23	24	21	38	42	115	249
233	260	101	43	30	20	24	24	29	44	98	256
234	292	74	38	6	1	12	23	14	67	74	244
235	236	102	28	39	27	27	27	16	30	119	253
236	245	81	49	31	29	15	30	31	36	98	245
237	239	97	40	31	26	28	25	25	43	90	256
238	241	97	38	15	24	20	23	25	46	111	245
239	230	103	38	40	32	27	31	38	41	102	246
240	243	92	32	66	24	19	30	24	37	104	238
241	242	101	38	30	22	28	25	24	53	98	248
242	234	105	35	21	30	23	19	23	43	89	228
243	238	85	50	36	20	21	29	23	41	90	232
244	234	81	45	29	19	20	23	24	47	99	231
245	235	91	50	29	22	28	30	28	38	103	231
246	239	106	24	50	11	24	17	17	32	101	227
247	223	103	32	34	21	20	21	25	48	105	220
248	230	84	32	26	29	25	28	34	48	85	230
249	220	87	46	6	33	23	19	20	49	73	222
250	213	72	32	34	20	26	21	29	42	82	230
251	232	86	35	33	16	18	21	27	40	117	228
252	211	86	33	24	23	26	23	27	55	94	218
253	223	86	49	28	26	22	20	25	38	90	224
254	229	108	38	45	30	24	23	29	42	88	223
255	215	95	30	22	30	16	38	44	46	85	216
256	229	71	22	32	21	19	15	23	40	102	232
257	232	86	35	27	24	19	19	30	38	92	227
258	220	104	34	25	15	13	24	25	45	87	208
259	217	64	48	22	21	20	25	28	39	94	228
260	216	80	29	30	35	16	24	33	45	81	232
261	225	81	41	21	29	24	29	35	56	100	219
262	218	86	33	25	26	29	23	28	36	82	216
263	221	83	39	29	14	25	19	20	40	82	223
264	213	87	42	34	22	29	31	32	30	84	216
265	207	94	41	26	22	23	16	34	39	95	215
266	223	70	37	20	22	16	23	24	29	79	215
267	204	79	36	33	31	24	34	29	47	79	211
268	238	86	33	19	25	13	23	14	25	45	231
269	210	86	46	21	27	23	20	27	39	90	218
270	193	75	31	26	22	21	31	28	35	87	199

Table 1. (Continued 5)

Sample Size (n)	p										
	0.05	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	0.95
271	230	152	9	4	3	4	16	9	60	98	264
272	215	86	46	24	21	26	15	26	39	76	201
273	177	75	57	34	31	0	24	14	23	85	200
274	209	77	43	20	23	22	16	29	37	91	207
275	212	77	44	31	23	22	20	31	34	87	208
276	215	86	36	35	28	27	24	28	38	109	194
277	212	77	32	21	24	15	23	30	31	81	203
278	200	66	40	32	24	16	22	31	33	89	204
279	196	70	41	20	21	19	13	31	42	73	202
280	209	74	38	18	29	17	25	20	36	97	192
281	212	67	37	25	27	21	27	24	36	101	219
282	206	72	37	32	32	25	25	27	36	81	203
283	185	75	29	17	21	30	20	28	38	73	176
284	183	78	30	33	25	38	24	28	40	75	205
285	196	93	35	31	21	32	20	23	41	67	198
286	197	77	35	15	30	30	32	32	31	86	188
287	189	69	41	31	21	24	12	34	34	73	193
288	190	81	49	34	40	31	14	31	32	74	200
289	180	83	43	27	20	32	18	36	37	83	196
290	184	68	36	33	26	27	21	24	42	87	198
291	197	85	29	25	18	22	31	23	41	66	185
292	209	87	38	22	13	30	24	13	46	18	231
293	198	65	27	30	22	32	29	37	42	78	182
294	203	72	24	29	16	21	19	36	43	88	186
295	203	67	27	24	19	27	22	32	50	85	185
296	196	57	45	27	17	13	17	24	27	81	192
297	182	71	28	22	24	23	25	31	40	73	175
298	215	69	43	30	24	21	15	25	44	68	184
299	180	92	20	36	27	15	25	23	30	74	197
300	**187**	**74**	**34**	**17**	**15**	**11**	**19**	**21**	**33**	**78**	**187**