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Signal Decimation Representation Associate with the Algebraic Signal Processing

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Abstract. Algebraic signal processing theory established the abstract frame for modern signal processing. In this paper, the algebraic representation of signal decimation in frequency domain for 1-D finite time model was studied on the basis of the algebraic signal processing theory. And the result was compared with existed results in classical signal processing. The results show that the algebraic signal processing theory is abstract representation and generalized form of classical theory for further.

Keywords: Fourier transform, polynomial algebras, shift, signal decimation, signal model

1 Introduction

The signal processing theory and technology have been widely used in high-tech fields such as aerospace, biomedicine, earth physics and so on. The research of the novel signal processing theory and technology has becomes one of the most important research topics in modern signal processing and applied mathematical society. The classical signal processing theory is based on the concepts of signal, filter (linear system), spectrum, and Fourier transform [1] and so on. However, with the rapid developments of the requirements of modern systems, we will not obtain the optimal results if we use the classical Fourier transform. For example, when processing the nonstationary signals or spare signals, the most useful tools are not the classical Fourier transform.

In order to overcome these shortcomings of the classical Fourier transform, the research of the novel signal processing tools for nonstationary signals and systems becomes one of the hottest research directions, many kinds of novel signal processing theory and method, for example the wavelet transform [2], fractional Fourier transform [3], linear canonical transformation [4] and time-frequency analysis [5], have been widely studied and play a good role in practical applications. Among them, the fractional domain signal processing methods [3-4, 6-8] and the sparse signal processing [9] methods receive much interests in recent years. The classical concepts and methods associate with the Fourier transform, for example, the sampling theorem, the correlation and convolution theorem and the uncertainty principles, are well investigated and studied in the fractional domain [10], one can find more results in fractional domain signal processing associated with the linear canonical transform in [4], the theories and applications of linear canonical transform in signal processing can be refer to the newly published book

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[4, 11].

It should be noted that all of the above mentioned signal processing methods are focus on the signal itself, and they have no ability of explore the overall structure of the whole signal spaces. In order to explore and study the overall structure of signal processing methods spaces and derive the novel signal processing methods, the application of the modern algebra and geometric theory to signal processing are becoming one of hottest research topics [12-18]. Among them, the algebraic signal processing (ASP) theory, which proposed by Puschel proposed and based on the classical signal processing methods [12-15, 17-18]. The core of the ASP is of linear signal model (A, M, Φ), where A is the filter algebra, and M is A-module that composed by signals, Φ is a bijective transform from vector space V to M. This signal model is relied on the definition of a shift operation, different shift operation will achieve different signal models, and there also exist the corresponding signal processing concepts, for example the filtering, spectrum and Fourier transform, associate with the different signal models. At the same time, beside the characterization and representation of signal processing methods based on the algebraic method, the research of the signal processing based on the modern theory of geometric theory and methods has becomes one of interesting topics in recent years [19-22]. Allof these methods can provide us a novel way for further exploration and study of signal processing methods from the overall structure of the signal spaces. From these theory we can not only explore the whole structure of these spaces, but also can propose more novel signal and information processing methods based on the derived results.

All of the signals processing in modern ASP theory are related to the discrete signal spaces, and in real application of discrete signal processing situations, we often encounter the problem of sampling a signal by decimation and interpolation. The decimation and interpolation is one of basic operations in multirate signal processing field [23]. The frequency effect of these two operations are well investigated in the classical Fourier transform domain, and there are two kinds of signal decimation: by time and frequency decimation. However, for the best of our knowledge, there are no paper published associated with the effect of. decimation and interpolation associate with the algebraic and geometric signal processing. Therefore, it has important and theoretical significance to study the representation of signal decimation in the framework of ASP.

In this paper, we investigate the characterization and representation of the signal decimation in the frequency domain. The paper is organized as following, Section 2 reviews the theory of the ASP, and Section 3 discusses the decimation in the framework of ASP, Section 4 is the conclusions of the paper.

2 Preliminaries

2.1 Algebraic Signal Processing

Algebraic signal processing (ASP) is a novel approach and generalization of linear signal processing that proposed by Puschel [13]. Where algebra refers to the theory of groups, rings, and fields of modern algebra. The classical concepts, for example, filtering, z-transform, spectrum, and Fourier transform of linear signal processing are all generalized to associated with ASP. It is shown in [24] that key observation underlying ASP is that standard signal processing is already algebraic in nature. The following example is proposed in [24].

$$A = h = \sum_{n \in Z} h_n x^n \mid h = (h_n)_{n \in Z} \in l^1(Z) \tag{1}$$

$$M = s = \sum_{n \in Z}^{n} s_n x^n \mid \mathrm{s} = (\mathrm{s}_n)_{n \in Z} \in l^2(Z) \tag{2}$$

For more results associate with the theory and methods of ASP, one can refer to [12-14, 24].

2.2 The Signal Model

From the theory of the algebraic signal processing, we can see that ASP is built on a signal model $(\mathrm{A}, \mathrm{M}, \phi)$ [24]. In general, we can conclude that the essence of ASP is to abstract the effect of filter to signal to filter space to signal space, and then the signal model is established based the algebraic operations of the signal space. The linear signal model can be rewritten as following [12].

Definition 1 (linear signal model [12]). A discrete linear signal model is defined as $(\mathrm{A},\mathrm{M},\phi)$, where A is a filter algebra, and M is M-module that composed by signals, let $V \le C^{I}$ is a complex signal on discrete set I, ϕ is a bijective transform from vector space V to M, we have $dim(V) = dim(M)$, and

$$\Phi : V \to M \tag{3}$$

and Φ is best explained through an example: With M as in Eq.(2),

$$\Phi : l^{2(Z)} \to M, s \mapsto \sum_{n\in Z} s_n x^{n}, \tag{4}$$

is the well-known z-transform [24].

After the definition of the signal model (A, M, Φ) it is therefore to ask which algebra and modules are used in ASP. The examples shown in [12] are associated with the infinite discrete-time signal processing. An important issue about this is which other algebra and modules actually occur in discrete signal processing and why. In order to better understand this we will start by understanding what' shift' and 'shift-invariance' means in our algebraic theory by focusing on the cases where only one shift is available in the following definition 2.

Definition 2 (Shift operator [12]). Shift operator x is a special filter, in other words, $x \in A$, in general, that is to say, every $h \in A$ can be represented as a polynomial or series of shift operator.

Mathematically, this means that the shift operator generates the algebra A. This kind of representation is an algebra in content of algebraic theory. All of the existing signal models, for example, the one-dimensional time, one-dimensional space, one-dimensional and two-dimensional spatial neighborhood signal model, are proposed in [12-14, 24], are established according to the different shift operations.

Another important concept in signal processing is shift-invariance. In the algebraic theory this property takes a very simple form. Namely, if x is the shift operator and a filter, then h is shift-invariant, if, for all signals s, $h(xs) = x(hs)$, which is equivalent to $hx = xh$. Requiring shift-invariance for all filters h thus means [24].

$$x{\cdot}h = h{\cdot}x, \text{for all } h \in A. \tag{5}$$

In general, we can establish a signal model by the following steps.

(1) Define the shift operation, for example, we can define the shift operation in time as $q \Diamond t_n = \mathrm{t}_{n+1}$, and q is the corresponding shift operator.

(2) Define the linear extension.

(3) Realization.

In one-dimensional discrete time space model, the shift operator is the conventional time shift operation, it is shown in [12-13] that the invariant signal model is equivalent to the exchangeable of algebra A. And the finite dimensional commutative algebra produced by x is actually a polynomial algebra, that is:

$$A = C[\mathrm{x}]/\mathrm{p}(\mathrm{x}) \tag{6}$$

where $\mathrm{p(x)}$ is a polynomial of order n, and $C[x]/p(x)$ is a set of polynomials which order are less than n in addition and modular multiplication.

2.2 The Fourier Transform in ASP

When the signal model is established, the corresponding concepts, for example, the filtering, spectrum, and the frequency response and the Fourier transform, are determined accordingly.

We assume that $p(x)$ is a separable polynomial, i.e.,

$$p(x)=\prod_{k=0}^{n-1}(x-a_k),\ \ a_k \neq \mathrm{a}_l, \quad \text{for} \quad k \neq l \quad \textbf{(7)}$$

We set $\alpha=(\alpha_0,....,\alpha_{n-1})$. In other words, separability means that p has no zeros of multiplicity larger than 1, and this property ensures that the spectrum of M consists exclusively of one dimensional spectral components.

The spectral decomposition of the regular module $M=C[x]/p(x)$ is also called the Fourier transform in ASP theory, and it is given by the Chinese remainder theorem as following [12].

$$\Delta: C[x]/\mathrm{p(x)} \rightarrow \mathrm{C[x]/(x-a_0)} \oplus ... \oplus C[x]/(\mathrm{x}-\alpha_{n-1}) \quad \textbf{(8)}$$

$$s \mapsto (\mathrm{s}(\alpha_0),...,s(\alpha_{n-1})) \quad \textbf{(9)}$$

If we let $C[x]/p(x)=M_k$, it is easy to see that M_k is of dimension 1, and the elements of $C[x]/p(x)$ are polynomials of degree of 0 or scalar. Further, M_k is an A-module. In this case, the above Fourier transform associate with the ASP can be rewritten as following.

$$\Delta: M \rightarrow \oplus\omega \in wM_\omega \quad \textbf{(10)}$$

$$s \mapsto (\mathrm{s}_\omega)\omega \in w \quad \textbf{(11)}$$

For finite shift invariant signal model, we have [14]

$$A=C[x]/p(x) \ =C[x]/(x^n-1) \quad \textbf{(12)}$$

Let $b=(1,\ x,\ ...),\ x^n-1$ is a basis of M M, then the Fourier transform associate this kind of model can be derived by Chinese remainder theorem as following.

$$\Delta: C[x]/(x^n-1) \rightarrow \oplus_{k=0}^{n-1} C[x]/(x-w_n^k) \quad \textbf{(13)}$$

$$s=s(x) \mapsto (s(w_n^0),\ s(w_n^0),...,\ s(w_n^{n-1})) \quad \textbf{(14)}$$

In this case, this kind of transform is just the classical discrete Fourier transform.

For more results about the ASP, one can refer to [5-8].

3 The Main Results

In multi-rate signal processing community, we often use signal decimation and interpolation operations to obtain the better results. Decimation and interpolation are most important operations in multi-rate signal process in fields. It is interesting and worthwhile to investigate this topics in context of ASP. In this section, we will investigate the signal decimation representations in the framework of algebraic signal processing theory.

In order to better show the derived results in this paper, we will consider two signal models in the framework of ASP theory. We let the parameters of signal model one as $A=M=C[x]/(x^n-1)$, and the basis of module as $b=(1,\ x,\ ...,\ x^{n-1})$; while the parameters of signal model two as

$A'=M'=C[x]/(x^{n/m}-1), n=km, k \in N^+$, and the corresponding basis of module M' is $b'=(1,\ x,\ ...,\ x^{n/m-1})$.

3.1 The Fourier Transform in ASP

Firstly, based on the definition of Fourier transform, we obtain the relationship between the Fourier transform under the different signal models.

Theorem 1. If we let the Fourier transform for a signal $s(x)$ in signal model one as $\Delta(s) = \left(s\left(\alpha_0,\ ...,\ s\left(\alpha_{n-1}\right)\right)\right)$, and the Fourier transform in signal model two as

$\Delta(s) = \left(s\left(\beta_0,\ ...,\ s\left(\beta_{n-1}\right)\right)\right)$, then we can obtain the relationship between them as following.

$$s(\beta_l) \ = s(\alpha_{ml}), 0 \le l \le n/m - 1. \qquad \textbf{(15)}$$

Proof. Without loss of generality, we take $m = 2$ as an example to prove this theorem, the other cases can be derived by the similar method. Suppose the representation of a signal $s(x)$ under the signal model one $A = M = C[x]/\left(x^{n-1}\right)$ as following.

$$s(\mathrm{x}) \ = \sum_{k=0}^{n-1} s_k x^k \bmod(x^n - 1) \qquad \textbf{(16)}$$

Then based on the Fourier transform definition, the Fourier transform ofsignal s(x) under the signal model one can be derived as:

$$s(\alpha_l) \ = \sum_{k=0}^{n-1} s_k e^{-j\frac{2\pi}{n}kl}, 0 \le l < n. \qquad \textbf{(17)}$$

By the similar method, we can derive the representation of s(x) under the signal model two $A' = M' = C[x]/\left(x^{n/2} - 1\right)$ as:

$$\begin{aligned} s(x) &= \sum_{k=0}^{n-1} s_k x^k \bmod(x^n - 1) = (\sum_{k=0}^{n/2-1} s_k x^k + \sum_{k=n/2}^{n-1} s_k x^k) \bmod(x^n - 1) \\ &= \sum_{k=0}^{n/2-1} (s_k + s_{n/2+k}) \mathrm{x}^k \bmod(x^n - 1) \end{aligned} \qquad \textbf{(18)}$$

At the same time, the corresponding Fourier transform of signal $s(x)$ under the signal model two as:

$$s(\beta_l) = \sum_{k=0}^{n/2-1} (s_k + s_{n/2+k}) \mathrm{e}^{-j\frac{4\pi}{n}kl}, 0 \le l < n/2 \qquad \textbf{(19)}$$

Based on the above equations, we can obtain that for $0 \le l < n/2$,

$$\begin{aligned} s(\alpha_{2l}) &= \sum_{k=0}^{n-1} s_k\, \mathrm{e}^{-j\frac{4\pi}{n}kl} = \sum_{k=0}^{n/2-1} s_k\, \mathrm{e}^{-j\frac{4\pi}{n}kl} + \sum_{k=n/2}^{n-1} s_k\, \mathrm{e}^{-j\frac{4\pi}{n}kl} \\ &= \sum_{k=0}^{n/2-1} s_k\, \mathrm{e}^{-j\frac{4\pi}{n}kl} + \sum_{m=0}^{n/2-1} s_{m+n/2}\, \mathrm{e}^{-j\frac{4\pi}{n}(m+n/2)l} \\ &= \sum_{k=0}^{n/2-1} (s_k + \mathrm{s(k+n/2)}) \mathrm{e}^{-j\frac{4\pi}{n}kl} \\ &= s(\beta_t) \end{aligned} \qquad \textbf{(20)}$$

This finishes the prove of theorem 1.

Theorem 1 shows the relationship between the Fourier transforms of the decimation operations. It should be noted that the above results are based on the definition of the signal model and the corresponding basis of the module M, if we change the basis of the module M′ in theorem 1 to

$$\vec{b} = \ (1,\ e^{-j\frac{2\pi}{n}x}, ..., e^{-j\frac{2\pi}{n}(n=m-1)\mathrm{x}^{n/m-1}}),$$

then we can obtain the following theorem 2.

Theorem 2. If we let the Fourier transform for a signal s(x) in signal model one as $\Delta(s)$ $=(s(\alpha_0),\ ...,\ s(\alpha_{n-1})$, and the Fourier transform in signal model two as $\Delta(s)=$ $(s(\gamma_0),\ ...,\ s(\gamma_{n/m-1}))$, then we can obtain the relationship between them as following.

$$s(\gamma_l)=s(\alpha_{ml+1}),0\le l\le n/m-1. \quad \textbf{(21)}$$

Proof. We also take $m = 2$ as an example to prove this theorem. Based on the method of proving the theorem 1, the Fourier transform of $s(x)$ under the signal model one is

$$\mathrm{s}(\alpha_l) = \sum_{k-0}^{n-1} s_k e^{-j\frac{2\pi}{n}kl}, 0\le l\le n. \quad \textbf{(22)}$$

At the same time, because the basis of the signal model two is changed to $\vec{b}=$ $(1,e^{-j\frac{2\pi}{n}x},...,e^{-j\frac{2\pi}{n}(n/m-1)x^{n/m-1}})$, therefore the representation of signals(x) under the signal model two is:

$$\mathrm{s(x)} = \sum_{k=0}^{n/2-1}(s_k - s_{n/2+k})x^k e^{-j\frac{2\pi}{n}k} \operatorname{mod}(x^{n/2-1}) \quad \textbf{(23)}$$

And the corresponding Fourier transform of s(x) under the signal model two as:

$$\mathrm{s}(\gamma_l) = \sum_{k=0}^{n/2-1}(s_k - s_{n/2+k})e^{-j\frac{2\pi}{n}k(2l+1)} = \mathrm{s}(\alpha_{2l+1}) \quad \textbf{(24)}$$

This finishes the proof of the theorem 2.

From the results of the above two theorems, it can be concluded that by choosing different basis of the module M one can realize the decimation of the discrete signal. It should be noted that when the basis and the module M are choose to be the classical one, the results derived in Theorem 1 and 2 becomes the classical results associated with the Fourier transform.

3.2 Examples

We propose the following two examples about the applications of the derived results for different signal models.

(1) In case of $n = 4,\ m = 2$.

In this case, the signal model one can be rewritten as $A=M=C[x]/(x^4-1)$, the basis of module M can derived as $\vec{b}=(1,x,x^2,x^3)$. The signal model two can be rewritten as $A'=M'=C[x]/(x^2-1)$, and the basis of module M′ is $\vec{b}=(1,x)$

Suppose the discrete signal is $s[n]=\delta[n-5]+\delta[n-3]+\delta[n+1]+\delta[n+2]$, then the signal s(x) can be represented under the signal model one as following

$$s(x) = x^{-2}+x^{-1}+x^3+x^5 \equiv x+x^2+\ 2x^3 mod(x^4-1) \quad \textbf{(25)}$$

Because the roots of $p(x)=x^4-1$ are $\alpha_2=e^{-j\frac{2\pi}{4}\times 2}=-1,\quad \alpha_0=1,\ \alpha_1=e^{-j\frac{2\pi}{4}}=-j,\quad \alpha_2=$ $e^{-j\frac{2\pi}{4}\times 2}=-1,\ \alpha_3=e^{-j\frac{2\pi}{4}\times 3}=j$ so the spectrum of discrete signal s [n] underthe signal model one can be derived as:

$$s(\alpha_0)=4, s(\alpha_1)=j-1, s(\alpha_2)=-2,\ s(\alpha_3)=-j-1. \quad \textbf{(26)}$$

By the similar method, one can derive the signal representation under thesignal model two as following

$$s(x) = x^{-2} + x^{-1} + x^3 + x^5 \equiv 1 + 3x \ mod(x^2 - 1) \tag{27}$$

And because the roots of $\mathrm{p(x) = x^2 - 1}$ are $\beta_0 = 1,\ \beta_1 = -1$, so the spectrum of discrete signal $s[n]$ under the signal model two can be derived as following.

$$s(\beta_0) = s(\alpha_0) = 4, s(\beta_1) = s(\alpha_2) = -2. \tag{28}$$

(2) In case of different basis of signal model.

In this case, we suppose the signal model two as $A' = M' = C[x]/(x^2 - 1)$, and the basis of module M' is $\left(1,\ \mathrm{e}^{-j\frac{\pi}{2}x}\right)$ then for the discrete signal the signal can be rewritten as following under the signal model two.

$$\begin{aligned} s(x) &= \mathrm{e}^{j\pi} x^{-2} + e^{j\frac{\pi}{2}} x^{-1} + e^{-j\frac{3\pi}{2}} x^3 + e^{-j\frac{5\pi}{2}} x^5 \\ &= -x^{-2} + jx^{-1} + jx^3 - jx^5 \\ &= -1 + jx \,\mathrm{mod}(x^2 - 1) \end{aligned} \tag{29}$$

And because the roots of $p(x) = x^2 - 2$ *are* $\gamma_0 = 1,\ \gamma_1 = -1$, therefor the spectrum of signal $s[n]$ under the signal model two can be derived as following.

$$s(\gamma_0) = -1 + j = s(\alpha_1),\ s(\gamma_1) = -1 - j = s(\alpha_3). \tag{30}$$

4 Conclusions

Based on the theory of the ASP and one-dimensional time signal model, the relationship between decimation operations based on ASP are investigated in details. The derived results can be seen as the generalization of the classical results to associate with the ASP. How to establish the connections of ASP with the fractional Fourier transform and linear canonical transform will be our future research directions.

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References

[1] A.V. Oppenheim, R. Schafer, J. Buck, Discrete-time signal processing, third ed., Prentice Hall, New York, 1999.

[2] I. Daubechied, Ten Lectures on Wavelets, Society for Industrial and Applied Mathematics, Philadelphia, 1992.

[3] R. Tao, B. Deng, Y. Wang, Fractional Fourier Transform and Its Applications, Tsinghua University Press, Beijing, 2009.

[4] T.Z. Xu, B.Z. Li, Linear Canonical Transform and Its Applications, Science Press, Beijing, 2013.

[5] L. Cohen, Time-frequency Analysis: Theory and Applications, Prentice Hall, New York, 1994.

[6] R. Tao, B.Z. Li, Y. Wang, On sampling of band limited signals associated with the linear canonical transform, IEEE Transactions on Signal Processing 56(11)(2008) 5454-5464.

[7] K.K. Sharma, S.D. Joshi, Signal separation using linear canonical and fractional Fourier transforms, Optics Communications 265(2)(2006) 454-460.

[8] B.Z. Li, Q.H. Ji, Sampling analysis in the complex reproducing kernel Hilbert space, European Journal of Applied Mathematics 26(01)(2015) 109-120.

[9] D.L. Donoho, Compressed sensing, IEEE Transactions on Information Theory 52(4)(2006) 1289C-1306.

[10] Q. Feng, B.Z. Li, Convolution theorem for fractional cosine-sine transform and its application, Mathematical Methods in the Applied Sciences 40(10)(2017) 3651-3665.

[11] J.J. Healy, H.M. Ozaktas, Linear Canonical Transforms: Theory and Applications, Springer, New York, 2016.

[12] M. Puschel, M. Rotteler, Algebraic signal processing theory: 2-D hexagonal spatial lattice, IEEE Transactions on Image Processing 16(6)(2007) 1506-1521.

[13] M. Puschel, J.F. Mura, Algebraic signal processing theory: foundation and 1-D time, IEEE Transactions on Signal Processing 56(8)(2008) 3572-3585.

[14] M. Puschel, J.F. Mura, Algebraic signal processing theory: 1-D space, IEEE Transactions on Signal Processing 56(8)(2008) 3586-3599.

[15] A. Sandryhaila, J. Kovacevic, M. Puschel, Algebraic signal processing theory: 1-D nearest-neighbor models, IEEE Transactions on Signal Processing 60(5)(2012) 2247-2259.

[16] A. Popoff, Fundamentals of Signal Processing in Metric Spaces with Lattice Properties, CRC Press, Boca Raton, FL, 2018.

[17] A. Sandryhaila, S. Saba, M. Puschel, Efficient compression of QRS complexes using Hermite expansion, IEEE Transactions on Signal Processing 60(2)(2012) 947-955.

[18] A. Sandryhaila, J. Kovacevic, M. Puschel, Algebraic signal processing theory: Cooley-Tukey type algorithms for polynomial transforms based on induction, SIAM Journal on Matrix Analysis and Applications 32(2)(2011) 364-384.

[19] R. Tao, B.Z. Li, H.F. Sun, Research progress of the algebraic and geometric signal processing, Defence Technology 9(1)(2013) 40-47.

[20] S. Amari, H. Nagaoka, Methods of Information Geometry (Translations of mathematical monographs), American Mathematical Society, New York, 2000.

[21] S. Amari, Differential-geometrical Methods in Statistics, Lecture Notes in Statistics, Springer-Verlag, Berlin, 1985.

[22] R. Nock, F. Nielsen, S. Amari, On conformal divergences and their population minimizers, IEEE Transactions on Information Theory 62(1)(2016) 527-538.

[23] R. E. Crochiere, L.R. Rabiner, Multirate Digital Signal Processing, Prentice Hall, New York, 1983.

[24] M. Puschel, Algebraic signal processing theory: an overview, in: Proc. Digital Signal Processing Workshop, IEEE Signal Processing Education Workshop, 2006.

[25] W.C. Curtis, I. Reiner, Representation Theory of Finite Groups and Associative Algebras, American Mathematical Society, Providence, 1962.