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A Multi-objective Evolutionary Algorithm for Fuzzy Mean-variance-entropy Portfolio Models with Transaction Cost and Liquidity

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Abstract. The main drawbacks of mean-variance model are to generate corner solutions and low diversity in the portfolios. To overcome these defects, firstly, we propose a new proportion entropy function as an objective function to generate well-diversified portfolio. Secondly, considering the transaction cost and liquidity, we present a new fuzzy mean-variance-entropy multi-objective portfolio selection model to find tradeoffs between risk, return and the diversification degree of portfolio, which is able to address a more realistic portfolio selection problem. Thirdly, we combined several efficient schemes to form an efficient algorithm to maintain the diversity of obtained solutions and to solve the presented multi-objective portfolio selection model. The proposed multi-objective portfolio model combined with the multi-objective evolutionary algorithm can overcome these defects fundamentally. Finally, to demonstrate the efficiency and effectiveness of the proposed model and algorithm, the designed algorithm is compared with two famous algorithms: multi-objective evolutionary algorithm based on decomposition (MOEA/D) and non-dominated sorting genetic algorithm II (NSGA-II) through some simulations based on the data of the Shanghai Stock Exchange Market. Simulation results show that the proposed algorithm is able to obtain better diversity and more evenly distributed Pareto fronts than the other two algorithms, and our proposed portfolio model can yield good performance of portfolio.

Keywords: entropy, fuzzy variable, multi-objective evolutionary algorithm, portfolio selection, possibilistic moments

1 Introduction

Modern portfolio selection theory is derived from the mean-variance (MV) model by Markowitz [1], which considered the trade-off between return and risk under stochastic environment by combining probability theory with optimization methods. After that, a lot of works on the mean-variance problem have been done in stochastic theory. However, such works have some problems in practice owing to the error in estimation of the moments of asset returns. The main issues in optimal mean-variance (MV) framework are that the portfolios are often extremely concentrated on a few assets or extreme positions, which are contradiction to the notion of diversification. The portfolio with low diversity may lead to loss while some of the invested assets experience unexpected gains, and makes the performance of the portfolio models not very good. Therefore, to generate a well-diversified portfolio, some researchers used the Shannon's entropy to measure the diversification in their portfolio selection model. For instance, Usta et al. [2] applied the Shannon's entropy as theoretical foundation to construct a mean-variance-skewness-entropy portfolio selection model, and obtained the good performance comparing with traditional

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portfolio models. Yu et al. [3] considered different entropy measures in the mean variance portfolio model, and enhanced the diversity of the portfolios. More recently, Ayusuk et al. [4] used the Shannon's entropy as the objective function to present a portfolio selection model based on mean-variance framework, and to find the optimal portfolio weights. However, these portfolio models with Shannon's entropy have serious issue, and the major issue of these portfolio models is that the Shannon's entropy measure is non-linear in portfolio selection since its weights take natural logarithm, which can result in very small positive weights for some asset in the portfolio [3]. As we known, in the practical investment management, such portfolios with very small positive weights are not effective for the purpose of diversity and are not what the investors really want. So using the Shannon's entropy to achieve diversification may contradict with the investor's notion of diversification. In order to overcome the shortcoming of the Shannon's entropy mentioned above, we present a novel proportion entropy function, which can directly acquire a diversified portfolio due to the linear feature. The detailed illustration of the novel proportion entropy function will be given in Section 3.

Notice that most of the existing portfolio selection models were based on probability theory under some random state. However, in financial markets, investors may face with imperfect information data and hence must deal with uncertain, imprecise and vague data. Furthermore, the uncertain returns of risk assets are influenced by economic change, politics, social conditions and the status of the related company, etc. In addition, security markets are usually very sensitive. An accident or a hard-to-verify message may influence security prices or returns greatly. Thus, in reality, the uncertain security returns are hard to get because of the complexity of the financial markets and many unexpected factors. Obviously, in many situations, it is more suitable to estimate the returns of risk assets by using fuzzy variables. In fact, with the development of security market and introduction fuzzy set theory, more and more researchers have investigated portfolio optimization problems by using fuzzy set theory, for example, Deng et al. [5] replaced the probabilistic mean, variance and covariance by the possibilistic mean, variance and covariance in mean-variance model, and used the gradually tolerant constraint method to solve the portfolio model. Nguyen et al. [6] proposed the fuzzy Sharpe ratio concept to evaluate the reward-to-volatility of portfolios in the fuzzy modeling context, and used the fuzzy approach and genetic algorithm to solve the model. More recently, Rubio et al. [7] used the weighted fuzzy time series methods to forecast the future performance of returns on portfolios, and modeled the uncertain parameters in the fuzzy portfolio selection model by using a possibilistic interval-valued mean approach. Calvo et al. [8] proposed a fuzzy portfolio selection model for dealing with the investor's subjective preferences about the desired return and acceptable risk properly, and used a nonlinear integer programming for solving the fuzzy portfolio selection problem. Additionally, Guo et al. [9] constructed a fuzzy multi-period portfolio selection problem under the credibility theory, and designed a fuzzy simulation method based on genetic algorithm for obtaining optimal solutions.

All of the above works mentioned mainly concern the risk-return trade-off. However, a reasonable investment decision requires considering many criteria simultaneously. In addition to risk and return of portfolio, the transaction cost and liquidity are very important factors for investors. As we know, ignoring transaction costs and liquidity would result in inefficient portfolios. Therefore, some researchers took into account the transaction cost or the liquidity in their portfolio selection models. For example, Mei et al. [10] analyzed the optimal portfolio policy for a multi-period mean-variance investor when he/she faced multiple risky assets in the presence of general transaction costs, they provided a closed-form expression for a no-trade region shaped as a parallelogram. Liu et al. [11] discussed the assets allocation in the presence of small proportional transaction costs and kept the asset portfolio close to a target portfolio when the trading cost is being reduced. Pae et al. [12] studied a log-robust portfolio optimization problem including transaction costs and developed an approximation method for the portfolio problem. Recently, Lin et al. [13] proposed a portfolio model with the minimum transaction lost, and used a fuzzy goal programming method for solving the portfolio model. But they did not consider the liquidity in their portfolio selection model. Although some researchers studied portfolio liquidity in the portfolio selection problems, they did not consider the diversification of portfolios in their models. For examples, Barak et al. [14] incorporated portfolio liquidity in their proposed model, and presented the liquidity as a trapezoidal fuzzy number and measured it by fuzzy credibility theory. Liu et al. [15] represented the liquidity of assets by interval variables and proposed a fuzzy multi-period portfolio optimization problem, and used the fuzzy decision-making theory to deal with the model. Gaigi et al. [16] researched a portfolio optimization problem under liquidity risk and price impact, and used a numerical approximation algorithm based on quantization procedure to solve the portfolio model. In practice, except for return, risk, transaction cost and liquidity, the diversification is one of the important principles of financial investment where investors should allocate their capital among different assets according to their own preferences. Therefore, the purpose of this paper is to increase the diversity of the portfolios and to overcome the shortcoming of Shannon's entropy by using our proposed proportion entropy function.

Although the great progress has been made in the previous studies, the vast majority portfolio selection problems focused on single objective portfolio selection model, which fails to meet the demand of investors who have multiple investment objectives. The main reason is that the multi-objective portfolio optimization problem cannot be easily solved because the challenge arises due to conflicting objectives, high occurrence of non-dominance of solutions based on the dominance relation, and under-diversification optimization solutions. On the other hand, in real investment environment, the investors usually have multiple conflicting and competing objectives, such as the total return, the total risk and the diversification degree of portfolio, to be optimized simultaneously and investors generally seek the best combination of assets among their investment objectives. Thus, in essence, the portfolio selection problem inherently involves multiple conflicting criteria. In problems of portfolio selection, financial decision makers explain objectives and investment purposes in the frame of multi-objective optimization problems which are more consistent with decision making realities. So, it is natural for us to construct the portfolio selection model in multi-objective framework. In order to achieve the better portfolio selection and set up a reasonable portfolio selection model, in this paper, in addition to consider the risk and return for portfolio selection, we also propose a new proportion entropy function as an objective function for the diversification of portfolio. To handle the imperfect information data and vague data, we introduce the fuzzy variable in the designed model. Furthermore, the transaction cost and liquidity are also considered in our model.

Since the proposed portfolio models are not differentiable, nonlinear and non-smooth multi-objective optimization problems (MOPs) which are very challenging and intractable optimization problems. From the viewpoint of optimization, a single solution that simultaneously optimizes all the conflicting objectives hardly exists in practice. Instead, there exists a set of acceptable efficient solutions (largely known as Pareto-optimal solutions or non-dominated solutions) which are optimal in such a way that no other solutions are superior to them when all objective are considered simultaneously. So, this demands us to find Pareto-optimal portfolios as many as possible. Among various multi-objective optimization algorithms, multi-objective evolutionary algorithms (MOEAs), which make use of the strategy of the population evolutionary to optimization the problems, are effective method for solving MOPs. In recent years, some famous MOEAs, such as non-dominated sorting genetic algorithm II (NSGA-II) [17], multi-objective evolutionary algorithm based on decomposition (MOEA/D) [18], a favorable weight-based evolutionary algorithm (FWEA) [19] and the p-optimality criteria evolutionary algorithm (p-OCEA) [20], have been proposed for solving the MOPs. In comparison with other traditional optimization algorithms, the MOEAs aim at finding a set of representative Pareto optimal solutions in a single run. Overall, the multi-objective optimization nature of the portfolio selection problem is unquestionable and the use of multi-objective optimization techniques has received a great deal of attention for solving these problems [21-22].

The main contributions of this study are as follows. Firstly, we present a novel linear proportion entropy function which can overcome the shortcomings of the Shannon's entropy function to generate more feasible diversified portfolios in fuzzy environment. Secondly, in order to obtain the diversity and convergence of solutions, we integrate several efficient schemes to form an efficient algorithm to solve the proposed model, and the numerical examples show that our proposed algorithm is able to obtain better diversity and more evenly distributed Pareto front than the classic MOEA/D and NSGA-II algorithms. Thirdly, our approach can find the Pareto front of efficient solutions that provide different Pareto-optimal investment strategies as diversified as possible for investors at a time, rather than one strategy for investors at a time. Finally, we proposed the possibilistic mean-variance-linear-proportion-entropy (MV-LPE) portfolio model that is more effective than the possibilistic mean-variance (P-MV) model and possibilistic mean-variavnce-Shannon's entropy (MV-SE) model.

The rest of this paper is organized as follows. Section 2 introduces basic definitions and preliminary results related to fuzzy variables. In Section 3, three weighted possibilistic multi-objective portfolio models with transaction cost and liquidity are presented. Section 4 presents a detailed description of our designed multi-objective evolution algorithm. Section 5 shows the comparisons of our algorithm with MOEA/D and NSGA-II, and gives the comparative result analysis of the three weighted possibilistic multi-objective portfolio models. The last Section gives some concluding remarks.

2 Preliminaries

We first introduce some basic definitions and results which will be needed in the following sections.

2.1 Fuzzy Numbers and Notations

In this section, the basic concepts and notations are given as follows [23-24]:

Definition 1. A fuzzy number $\tilde{A}$ is described as any fuzzy subset of the real line $\mathbb{R}$, whose membership function $\mu_{\tilde{A}}:\mathbb{R}\to[0,1]$ satisfies the following conditions:

(i) $\tilde{A}$ is normal, i.e., there exists an $x\in\mathbb{R}$ such that $\mu_{\tilde{A}}(x)=1$,

(ii) $\mu_{\tilde{A}}(x)$ is upper semi-continuous, i.e., $\mu_{\tilde{A}}\left(\lambda x+(1-\lambda)y\right)\le\min\left\{\mu_{\tilde{A}}(x),\mu_{\tilde{A}}(y)\right\}$, for all $\lambda\in[0,1]$,

(iii) $\mu_{\tilde{A}}(x)$ is upper semi-continuous, i.e., $\left\{x\in\mathbb{R}\middle|\mu_{\tilde{A}}(x)\le\varepsilon\right\}$ is a closed set, for all $\varepsilon\in[0,1]$,

(iv) The closure of the set $\left\{x\in\mathbb{R}\middle|\mu_{\tilde{A}}(x)>0\right\}$ is a compact set.

Definition 2. A γ - level set of $\tilde{A}$ is defined by an ordinary set $\tilde{A}_\gamma=\left\{x\in\mathbb{R}\middle|\mu_{\tilde{A}}(x)\ge\gamma\right\}$ for $\gamma\in[0,1]$, if $\gamma>0$ and $\tilde{A}_\gamma=\mathrm{cl}\left\{x\in\mathbb{R}\middle|\mu_{\tilde{A}}(x)\ge 0\right\}$ (the closure of the support of $\tilde{A}$) if $\gamma=0$. As well known, if $\tilde{A}$ is a fuzzy number, then $\tilde{A}_\gamma=\left\{x\in\mathbb{R}\middle|\mu_{\tilde{A}}(x)>\gamma\right\}=\left[\underline{a}(\gamma),\overline{a}(\gamma)\right]$ is a compact subset of $\mathbb{R}$ for all $\gamma\in[0,1]$.

Definition 3. A fuzzy number $\tilde{A}$ is called a trapezoidal fuzzy number with core $[c,d]$, left width $\delta>0$ and right width $\theta>0$, if its membership function has the following form:

$$\mu_{\tilde{A}}(x)=\begin{cases}1-\dfrac{c-x}{\delta}, & \text{if } c-\delta\le x\le c\\ 1, & \text{if } c\le x\le d\\ 1-\dfrac{x-d}{\theta}, & \text{if } d\le x\le d+\theta\\ 0, & \text{if otherwise}\end{cases} \tag{1}$$

and it can be denoted by $\tilde{A}=\left(c,d,\delta,\theta\right)$.

2.2 Weighted Possibilistic Moments (WPS)

In this paper, we regard a new weighted possibilistic mean value and variance [25] of fuzzy variables as the return level and risk level, respectively. It is just because the weighted possibilistic mean (WPM) and weighted possibilistic variance (WPV) of the fuzzy numbers have all the properties of the possibilistic mean value and variance stated in reference [23], and WPV has all necessities and important properties for defining the possibilistic variance of a fuzzy number. In addition, WPM is the nearest weighted point to the fuzzy number via minimizing a new weighted distance quantity, moreover, WPV of a fuzzy number is consistent with the physical interpretation of the variance and well-known definition of variance in probability theory so that it can simply introduce the possibilistic moments about the mean of fuzzy numbers without problem [25]. Furthermore, Pasha et al. [25] pointed out that the definition of weighted possibilistic moments on fuzzy numbers is more suitable for all fuzzy numbers than the definitions of possibilistic moments introduced in reference [23-24].This indicates that WPM and WPV are suitable and applicable to measure the risk and return, and play an important role in fuzzy data analysis. For this reason, we quantify the risk and the return by using WPV and WPM respectively.

The definitions of the weighted possibilistic moments of fuzzy numbers are given in [25] as follows.

Definition 4. Let $\tilde{A}$ be a fuzzy number with $\tilde{A}_\gamma = [\underline{a}(\gamma), \overline{a}(\gamma)]$, $\gamma \in [0,1]$. Then WPM (or the first order f-WPM) and WPV of fuzzy number $\tilde{A}$ are defined as follow, respectively.

(i) $M_f(\tilde{A}) = \int_0^1 f(\gamma)\dfrac{\underline{a}(\gamma)+\overline{a}(\gamma)}{2}d\gamma$,

(ii) $Var_f(\tilde{A}) = \dfrac{1}{2}\int_0^1 f(\gamma)\left[\left(\underline{a}(\gamma)-M_f(\tilde{A})\right)^2 + \left(\overline{a}(\gamma)-M_f(\tilde{A})\right)^2\right]d\gamma$.

where, $f(\gamma) = (n+1)\gamma^n$ is a weighted function such that $\int_0^1 f(\gamma)d\gamma = 1$.

Definition 5. Let $\tilde{A}$ and $\tilde{B}$ be two fuzzy numbers with possibilistic mean $M(\tilde{A})$ and $M(\tilde{B})$, respectively. Then the weighted possibilistic covariance between $\tilde{A}$ and $\tilde{B}$ is defined as

$$Cov_f\left(\tilde{A},\tilde{B}\right) = \frac{1}{2}\int_0^1 f(\gamma)\left[\left(\underline{a}(\gamma)-M_f(\tilde{A})\right)\left(\underline{b}(\gamma)-M_f(\tilde{B})\right) + \left(\overline{a}(\gamma)-M_f(\tilde{A})\right)\left(\overline{b}(\gamma)-M_f(\tilde{B})\right)\right]d\gamma,$$

where, $f(\gamma) = (n+1)\gamma^n$ is a weighted function such that $\int_0^1 f(\gamma)d\gamma = 1$.

Theorem 1. [25] Let $\tilde{A}$ and $\tilde{B}$ be two fuzzy numbers, and let λ and μ be nonnegative numbers. Then the following conclusions can be obtained:

(i) $M_f\left(\lambda\tilde{A} \pm \mu\tilde{B}\right) = \lambda M_f(\tilde{A}) \pm \mu M_f(\tilde{B})$,

(ii) $Var_f\left(\lambda\tilde{A} \pm \mu\tilde{B}\right) = \lambda^2 Var_f(\tilde{A}) + \mu^2 Var_f(\tilde{B}) \pm 2\lambda\mu Cov_f\left(\tilde{A},\tilde{B}\right)$.

From Theorem 1 above, we can easily deduce the following theorem.

Theorem 2. Let $\tilde{A}_1$, $\tilde{A}_2$,…, $\tilde{A}_n$ be n fuzzy numbers, and let λ_1, λ_2,…, λ_n be n nonnegative real number. Then

(i) $M_f\left(\sum_{i=1}^n \lambda_i\tilde{A}_i\right) = \sum_{i=1}^n \lambda_i M_f\left(\tilde{A}_i\right)$,

(ii) $Var_f\left(\sum_{i=1}^n \lambda_i\tilde{A}_i\right) = \sum_{i=1}^n \lambda_i^2 Var_f\left(\tilde{A}_i\right) \pm 2\sum_{i=1}^n\sum_{j=1}^n \lambda_i\lambda_j Cov_f\left(\tilde{A}_i,\tilde{A}_j\right)$.

2.3 Multi-objective Optimization Problems (MOPs)

In this paper, the proposed portfolio selection problems will be modeled as multi-objective optimization problems (MOPs).Typically, a multi-objective optimization problem can be formulated as follows [26]:

$$\begin{cases} \min \quad y = F(x) = [f_1(x), f_2(x), \cdots, f_k(x)] \\ s.t \;\; g_j(x) \ge 0, \; j = 1,2,\cdots,q \\ \qquad h_j(x) = 0, \; j = q+1,\cdots,m \end{cases} \tag{2}$$

The feasible region Ω is defined as follows:

$$\Omega = \left\{x \middle| g_j(x) \ge 0, j = 1,2,\cdots,q; h_j(x) = 0, \; j = q+1,\cdots,m\right\} \subseteq S \subseteq R^n. \tag{3}$$

Definition7 (Pareto-Dominance). A solution $x^* = \left(x_1^*, x_2^*, \cdots, x_n^*\right)$ is said to dominate (Pareto-optimal) another solution $x = \left(x_1, x_2, \cdots, x_n\right)$ (denoted by $x^* \succ x$), if the following conditions are satisfied:

(i) $\forall j \in \{1,2,\cdots,k\} : f_j(x^*) \le f_j(x)$,

(ii) $\exists j \in \{1,2,\cdots,k\} : f_j(x^*) < f_j(x)$.

Definition 8 (Pareto-Optimal). A solution $x^* \in \Omega$ is said to be non-dominated (Pareto-optimal) iff $\neg\exists x$ such that $x \succ x^*$.

Definition 9 (Pareto-Optimal Set). The set of all Pareto optimal solutions is defined as:

$$PS = \left\{ x^* \middle| \neg\exists x \in \Omega : x \succ x^* \right\}. \quad \textbf{(4)}$$

Definition 10 (Pareto-Optimal Front). The set of all Pareto solutions in objective space is defined as Pareto front and denoted as follow:

$$PF = \left\{ \left[f_1(x), f_2(x), \cdots, f_k(x) \right] \middle| x \in PS \right\}. \quad \textbf{(5)}$$

3 Weighted Possibilistic Multi-objective Portfolio Selection Models

In this section, we first introduce the problem description and notations used. Second, we propose a novel proportion entropy for diversifying the allocation on various assets. Then, based on the new weighted possibilistic moments of fuzzy variable, we set up the fuzzy mean-variance-entropy multi-objective portfolio models.

3.1 Problem Description and Notations

Let us consider a multi-objective fuzzy portfolio selection problem with n risk assets. The return rates and turnover rates of the risk assets are denoted as trapezoidal fuzzy numbers. For the notation convenience, we introduce the following notations.

x_i : Proportion of the total investment devoted to the risk asset i, $i = 1, 2, \cdots, n$,

k_i : Rate of transaction cost on the risk asset i, $i = 1, 2, \cdots, n$,

$\tilde{r}_i$: Fuzzy rate of return on the risk asset i, $i = 1, 2, \cdots, n$,

$\tilde{l}_i$: Fuzzy turnover rate of the risk asset i, $i = 1, 2, \cdots, n$.

In this paper, the transaction cost is assumed to be a V-shape function, which is the difference between a new portfolio $x = (x_1, x_2, \cdots, x_n)$ and a given portfolio $x^0 = (x_1^0, x_2^0, \cdots, x_n^0)$. For a new investor, it can be argued that $x_i^0 = 0$, $i = 1, 2, \cdots, n$. Thus the total transaction cost of the portfolio is $\sum_{i=1}^{n} k_i \left| x_i - x_i^0 \right|$.

For any risk asset, liquidity may be measured by using the turnover rate which is defined by the ratio of the average trading volume of the assets trade in the market and the trading volume of the tradable asset (i.e., shares held by the public) corresponding to the asset. It is well known that the future turnover rates of assets cannot be accurately predicted in the uncertain financial market. Therefore, the fuzzy set theory provides a new tool to deal with this imprecision. Without loss of generality, the turnover rate of the ith asset is regarded as a trapezoidal fuzzy number $\tilde{l}_i = (c_i, d_i, \delta_i, \theta_i)$, $i = 1, 2, \cdots, n$ in this paper. Note that the turnover rate of the portfolio $x = (x_1, x_2, \cdots, x_n)$ is $\tilde{l}(x) = \sum_{i=1}^{n} \tilde{l}_i x_i$. In the model, the portfolio liquidity is always greater or equal to a given tolerate level of fuzzy turnover rate $\tilde{l}_0$ by the investor, that is, $\tilde{l}(x) = \sum_{i=1}^{n} \tilde{l}_i x_i \geq \tilde{l}_0$. According to the method for ranking fuzzy number reported in reference [27], the fuzzy inequality $\sum_{i=1}^{n} \tilde{l}_i x_i \geq \tilde{l}_0$ can be transformed into the crisp in-equality $M\left(\sum_{i=1}^{n} \tilde{l}_i x_i \right) \geq M\left(\tilde{l}_0 \right)$.

3.2 Proportion Entropy for Decentralized Investment

In order to satisfy the requisition of decentralized investment, a new linear proportion entropy will be proposed to enhances diversity of the portfolios. Before introducing the proportion entropy, let us first review the Shannon's entropy, which is employed to reflect the diversification degree of the aforementioned portfolio selection problem in literatures [2-3]. And its mathematical expression can be expressed as follows:

$$\mathrm{Sn}(x) = -\sum_{i=1}^{n} x_i \ln x_i \tag{6}$$

where $x_i > 0$ denotes as the investment proportion of asset i, $i = 1,2,\cdots,n$. As pointed out by Yu et al. [3], the Shannon's entropy measure is non-linear in portfolio selection as its weights take natural logarithm, which leads to very small positive weights for some asset in the portfolio. However, in the practical investment management, such portfolios with small positive weights are not effective for the purpose of diversity and are not what the investors really want. So using the Shannon's entropy to assess diversification may contradict with the investor's notion of diversification. In order to overcome the shortcoming of the Shannon's entropy mentioned above, we present a novel proportion entropy function which aims to minimize the sum of distance between weights of the invested asset. The following is the definition of our proportion entropy:

$$\mathrm{Pn}(x) = -\left(\sum_{i=1}^{n-1} \left|x_i - x_{i+1}\right|^z\right)^z \tag{7}$$

where, z is a constant and $z \geq 1$. From the Eq. (7), as can be seen that the more equally the budget allocates on assets, the larger the proportion entropy is. It is worth noticing that the proportion entropy can be transferred into a linear type when $z = 1$. The following is the linear proportion entropy type as $z = 1$.

$$\mathrm{Pn}(x) = -\sum_{i=1}^{n-1} \left|x_i - x_{i+1}\right| \tag{8}$$

The linearization of the proportion entropy can directly result in a diversified portfolio because of the linear feature of the proportion entropy [3]. By using our proposed linear proportion entropy in portfolio selection model, in essence, we can diversify the allocation on various assets, which meets the requirement of investors.

3.3 Construction of Fuzzy Multi-objective Portfolio Selection Models

For a rational investor, who wants to maximize return after paying transaction costs and minimize the risk of portfolio. Based on the above discussions, a fuzzy portfolio selection model with transaction costs and fuzzy liquidity constraints can be described as the following weighted possibilistic mean-variance (P-MV) model:

$$\begin{cases} \max \quad M_f\left[\sum_{i=1}^{n} \tilde{r}_i x_i\right] - \sum_{i=1}^{n} k_i \left|x_i - x_i^0\right| \\ \min \quad V_f\left[\sum_{i=1}^{n} \tilde{r}_i x_i\right] \\ s.t \quad M\left(\sum_{i=1}^{n} \tilde{l}_i x_i\right) \geq M\left(\tilde{l}_0\right) \\ \qquad \sum_{i=1}^{n} x_i = 1,\ x_i \geq 0,\ i = 1,2,\cdots,n \end{cases} \tag{9}$$

where the first constraint means the liquidity is greater than or equal to a given value $\tilde{l}_0$ by the investor, and the second constraint implies that all the capitals will be invested to n assets and short-selling is not allowed.

To measure the diversification degree, we introduce our linear proportion entropy to the weighted possibilistic mean variance model and obtained the following three-objective weighted possibilistic mean-variance-linear-proportion-entropy (MV-LPE) model:

$$\begin{cases} \max \quad M_f\left[\sum_{i=1}^{n}\tilde{r}_i x_i\right]-\sum_{i=1}^{n}k_i\left|x_i-x_i^0\right| \\ \min \quad V_f\left[\sum_{i=1}^{n}\tilde{r}_i x_i\right] \\ \min \quad \mathrm{Pn}(x)=\sum_{i=1}^{n-1}\left|x_i-x_{i+1}\right| \\ s.t \quad M\left(\sum_{i=1}^{n}\tilde{l}_i x_i\right)\ge M\left(\tilde{l}_0\right) \\ \qquad \sum_{i=1}^{n}x_i=1,\ x_i\ge 0,\ i=1,2,\cdots,n \end{cases} \tag{10}$$

Corollary1. Let $\tilde{r}_i=(a_i,b_i,\alpha_i,\beta_i)$ and $\tilde{l}_i=(c_i,d_i,\delta_i,\theta_i)$ represent the fuzzy returns and the fuzzy turnover rates of the *i*th asset for $i=1,2,\cdots,n$ and $\tilde{l}_0=(c_0,d_0,\delta_0,\theta_0)$ be a given fuzzy turnover rate confidence level by the investor. Then the model (10) can be converted into the following deterministic MOPs (11).

$$\begin{cases} \max \quad f_1(x)=\left[\sum_{i=1}^{n}\left(\frac{a_i+b_i}{2}+\frac{\beta_i-\alpha_i}{6}\right)x_i\right]-\sum_{i=1}^{n}k_i\left|x_i-x_i^0\right| \\ \min \quad f_2(x)=\frac{\left[\sum_{i=1}^{n}x_i\left(\beta_i+\alpha_i\right)\right]^2+\left[\sum_{i=1}^{n}x_i\left(\beta_i-\alpha_i\right)\right]^2}{72}+\left[\sum_{i=1}^{n}x_i\left(\frac{b_i-a_i}{2}+\frac{\beta_i+\alpha_i}{6}\right)\right]^2 \\ \max \quad f_2(x)=-\sum_{i=1}^{n-1}\left|x_i-x_{i+1}\right| \\ s.t \quad g_1(x)=\sum_{i=1}^{n}\left(\frac{c_i+d_i}{2}+\frac{\theta_i-\delta_i}{6}\right)x_i-\left(\frac{c_0+d_0}{2}+\frac{\theta_0-\delta_0}{6}\right)\ge 0 \\ \qquad h_1(x)=\sum_{i=1}^{n}x_i=1,\ x_i\ge 0,\ i=1,2,\cdots,n \end{cases} \tag{11}$$

Proof. Since $\tilde{r}_i=(a_i,b_i,\alpha_i,\beta_i)$ and $\tilde{l}_i=(c_i,d_i,\delta_i,\theta_i)$ are trapezoidal fuzzy numbers for $i=1,2,\cdots,n.$ According to the Extension Principle of fuzzy sets, $\sum_{i=1}^{n}\tilde{r}_i x_i=\left(\sum_{i=1}^{n}x_i a_i,\sum_{i=1}^{n}x_i b_i,\sum_{i=1}^{n}x_i\alpha_i,\sum_{i=1}^{n}x_i\beta_i\right)$ and $\sum_{i=1}^{n}\tilde{l}_i x_i=\left(\sum_{i=1}^{n}x_i c_i,\sum_{i=1}^{n}x_i d_i,\sum_{i=1}^{n}x_i\delta_i,\sum_{i=1}^{n}x_i\theta_i\right)$ are also trapezoidal fuzzy numbers for $i=1,2,\cdots,n.$ Combining this with Theorem 1 and Theorem 2, we can get this conclusion.

In order to highlight the advantage of our designed linear proportion entropy over the Shannon's entropy as the diversification measure for multi-objective portfolio, the weighted possibilistic mean-variavnce-Shannon's entropy (MV-SE) model is given as follows:

$$\begin{cases} \max & f_1(x)=\left[\sum_{i=1}^{n}\left(\frac{a_i+b_i}{2}+\frac{\beta_i-\alpha_i}{6}\right)x_i\right]-\sum_{i=1}^{n}k_i\left|x_i-x_i^0\right| \\ \min & f_2(x)=\frac{\left[\sum_{i=1}^{n}x_i\left(\beta_i+\alpha_i\right)\right]^2+\left[\sum_{i=1}^{n}x_i\left(\beta_i-\alpha_i\right)\right]^2}{72}+\left[\sum_{i=1}^{n}x_i\left(\frac{b_i-a_i}{2}+\frac{\beta_i+\alpha_i}{6}\right)\right]^2 \\ \max & f_2\left(x\right)=-\sum_{i=1}^{n}x_i\ln x_i \\ s.t & g_1(x)=\sum_{i=1}^{n}\left(\frac{c_i+d_i}{2}+\frac{\theta_i-\delta_i}{6}\right)x_i-\left(\frac{c_0+d_0}{2}+\frac{\theta_0-\delta_0}{6}\right)\geq 0 \\ & h_1(x)=\sum_{i=1}^{n}x_i=1,\ x_i\geq 0,\ i=1,2,\cdots,n \end{cases} \tag{12}$$

4 A Multi-objective Evolutionary Algorithm for Portfolio Selection

In this section, we will describe the designed algorithm. To maintain the diversity and good convergence of the obtained solutions, the proposed algorithm is specifically designed and is included the four parts, i.e. space and population decomposition, update strategy, crossover operations and selection strategy. Space and population decomposition and update strategy are used for keeping the diversity of the obtained solutions. The crossover operations are used to search the decision space. The selection strategy can make the obtained solutions converge to the PF .

4.1 Space and Population Decomposition

We use a set of direction vectors to decompose the objective space of these multi-objective portfolio models into a set of sub-objective space and use these direction vectors to classify the obtained solutions to make each sub-objective space own a solution [28]. For a given set of direction vectors $\left(\lambda^1,\lambda^2,\cdots,\lambda^N\right)$, and the set of current obtained solutions is represented by the symbol POP , these solutions will be classified by the following formula [28]:

$$\begin{gathered} P^i=\left\{x\middle|x\in POP,\Delta\left(F(x),\lambda^i\right)=\max_{1\leq j\leq N}\left\{\Delta\left(F(x),\lambda^j\right)\right\}\right\} \\ \Delta\left(F(x),\lambda^i\right)=\frac{\lambda^i*\left(F(x)-Y\right)^T}{\left\|\lambda^i\right\|*\left\|F(x)-Y\right\|},\ i=1,2,\cdots,n \end{gathered} \tag{13}$$

where $Y=\left(Y_1,Y_2,\cdots,Y_m\right)$ is a reference point and $Y_i=\min\left\{f_i(x)\middle|x\in\Theta\right\}$, $\Delta\left(F(x),\lambda^i\right)$ is the cosine of the angle between λ^i and $F(x)-Y$. These solutions are divided into M classes by the formula (13) and the objective space Θ is divided into M sub-objective spaces $\Theta_1,\Theta_2,\cdots,\Theta_M$.

$$\text{Where } \Theta_k\left(k=1,2,\cdots,M\right)\text{ is } \Theta_k=\left\{F(x)\middle|x\in\Theta,\Delta\left(F(x),\lambda^k\right)=\max_{1\leq j\leq N}\left\{\Delta\left(F(x),\lambda^j\right)\right\}\right\} \tag{14}$$

If a $P^i\left(1\leq i\leq M\right)$ is empty, a solution is randomly selected from POP and put into P^i . We could make the following comments on the classification (decomposition) method [28]:

(i) This method is equivalent in a way that PF of all these sub-objective spaces made up of PF of the problems PMV, MV-SE and MV-LPE.

(ii) Even if the PS of these problems has a nonlinear geometric shape, the PS of each sub-objective space could be approaching linear, because it is just a small part of the PS of (5). Thus, formulas (13) and (14) make the problems PMV, MV-SE and MV-LPE are simpler than before, at least in accordance to with PS shapes.

(iii) This classification (decomposition) method does not need any aggregation methods. A user only needs to select a set of direction vectors. To some extent, it requires little human labor.

4.2 Update Strategy for Decomposition of POP

The elitist strategy generally can increase the speed of the convergence and improve the solution quality. A new elitist strategy is designed in our algorithm. The detail is as follows. For each sub-region of POP, there are two cases:

(1) If the sub-region is empty, the solution is selected and retained, whose objective vector has the smallest angle to the direction vector λ^i corresponding to this sub-region.

(2) If the sub-region is not empty, there exist two situations. If this sub-region has only one solution, the solution is retained. If the number of the solutions in this sub-region is equal to or greater than two, the non-dominated solutions are the candidate ones to be chosen. Then the non-dominated solution is selected and retained, whose objective vector has the smallest angle to the direction vector λ^i corresponding to this sub-region.

4.3 Crossover Operations and Selection Strategy

In our algorithm, we use a differential evolution (DE) [29] operator and the quantization orthogonal crossover (QOX) as the crossover operations. The crowding distance [17] is used to calculate the fitness value of a solution in the selection operators to improve the convergence.

4.4 Steps of the Proposed Algorithm

Based on the above discussed, a new orthogonal evolutionary algorithm based on Decomposition is proposed. The steps of our algorithm are as follows:

Step 1. (Initialization) Given M direction vectors $\left(\lambda^1,\lambda^2,\cdots,\lambda^M\right)$, randomly generate an initial population $POP(k)$ and its size is M, let $k=0$. Set $Y_i=\min\left\{f_i(x)\middle|x\in POP(k)\right\}$, $1\le i\le m$.

Step 2. (Fitness) Solutions of $POP(k)$ are firstly divided into M classes by the formula (13) and the fitness value of each solution in $POP(k)$ is calculated by using the crowding distance. Then some better solutions are selected from the population $POP(k)$ and put into the population POP. In this paper, binary tournament selection is used.

Step 3. (New solutions) Apply genetic operators and QOX to the parent population to generate offspring. The set of all these offspring is denoted as O.

Step 4. (Update) Y is firstly updated. For each $j=1,2,\cdots,m$, if $Y_j>\min\left\{f_j(x)\middle|x\in O\right\}$, then set $Y_j=\min\left\{f_j(x)\middle|x\in O\right\}$. The solutions of $POP(k)\cup O$ are firstly classed by the formula (13), then M best solutions are selected by the update strategy of Section 4.2 and put into $POP(k+1)$. Let $k=k+1$.

Step 5. (Termination) If stop condition is satisfied, stop; otherwise, go to Step 2.

5 Numerical Examples and Analysis

In this section, in order to demonstrate the effectiveness of the modeling idea and the proposed algorithm, we first compare our algorithm with two famous algorithms: NSGA-II [17] MOEA/D [18] through some numerical examples based on the data of the Shanghai Stock Exchange Market. Then, we use the proposed algorithm to solve these fuzzy multi-objective portfolio models to illustrate the idea of our model and the effectiveness of the designed algorithm. Finally, the comparison analysis for these portfolio models will be given by these numerical examples.

5.1 Data Processing

We assume that an investor chooses 12 assets from Shanghai Stock Exchange for his or her investment. The exchange codes of the 12 assets are 601098, 601880, 600563, 600038, 601888, 601377, 600721, 600681, 600571, 600419,600570, 600201. For convenience of description, we denote the 12 assets successively as Assets 1,2,3,4,5,6,7,8,9,10,11,12 in these examples. Here, we collect original data from the weekly closing pricing and turnover rate of these assets in three years from January 2012 to January 2015 as the sample data. Using the simple estimation method in Vercher et al. [30], we deal with the historical data of 12 rates. Then, the parameters of the fuzzy return rate $\tilde{r}_i = (a_i, b_i, \alpha_i, \beta_i)$ and the turnover rates $\tilde{l}_i = (c_i, d_i, \delta_i, \theta_i)$ of the ith asset are shown respectively in Table 1 and Table 2.

Table 1. Fuzzy returns of 12 assets

Asset ***i***	1	2	3	4
Return rate	(-0.0065,0.0188 0.0562 0.0681)	(-0.0094,0.0077 0.0532 0.1034)	(-0.0062,0.0219 0.0736 0.0694)	(-0.0076,0.0154 0.0633 0.0797)
Asset ***i***	5	6	7	8
Return rate	(-0.0047,0.0144,0.0562,0.0660)	(-0.0093,0.0189,0.0711,0.0954)	(-0.0053,0.0180,0.0814,0.0795)	(-0.0044,0.0228,0.0817,0.0764)
Asset ***i***	9	10	11	12
Return rate	(-0.0029,0.0271, 0.0830,0.0906)	(-0.0124,0.0199, 0.0744,0.1258)	(0.0063,0.0139, 0.0637,0.1076)	(-0.0060,0.0193, 0.0511,0.0746)

Table 2. Fuzzy turnover rates of 12 assets

Asset ***i***	1	2	3	4
Return rate	(0.0416,0.0662,0.0224,0.1315)	(0.0434,0.0639,0.0352,0.2148)	(0.0526,0.0657,0.0290,0.0599)	(0.0508,0.0723,0.0338,0.0994)
Asset ***i***	5	6	7	8
Return rate	(0.0220,0.0278,0.0124,0.0571)	(0.0449,0.0699,0.0239,0.1900)	(0.0723,0.0990,0.0481,0.1264)	(0.0708,0.0954,0.0426,0.1297)
Asset ***i***	9	10	11	12
Return rate	(0.0499,0.0820,0.0315,0.0965)	(0.0705,0.0970,0.0515,0.2344)	(0.0299,0.0503,0.0194,0.0875)	(0.0290,0.0379,0.0164,0.0590)

In our numerical examples, we assume that the transaction costs of assets are identical, i.e., $k_i = 0.003$ for all $i = 1,2,\cdots,12$, and the fuzzy turnover rate confidence level given by the investor is $\tilde{l}_0 = (0.0227, 0.0322, 0.0658, 0.2324)$.

5.2 Parameter Settings

Three compared algorithms, i.e., our algorithm, NSGA-II and MOEA/D, are implemented on a personal computer (Intel Xeon CPU 2.53GHz, 4.0G RAM). The individuals are all coded as the real vectors. Polynomial mutation [31] operators are applied directly to real parameter values in three compared algorithms. For crossover operators, simulated binary crossover (SBX [31]) is used in NSGAII, the DE and QOX are used in our algorithm, and the DE is also used in MOEA/D. The population size is 105 in three algorithms. Weight vectors are generated by using the method which is used in [18]. The direction vectors of our algorithm are the weight vectors. The number of weight vectors is equal to the population size. The number of the weight vectors in the neighborhood in MOEA/D is 20 for all numerical examples. The Tchebycheff approach [18] is used for MOEA/D as an aggregate function. Other control parameters in MOEA/D are the same as in [18]. Quantization levels $Q = 3$ and there are $F = 3$ factors in the QOX operator. So, the each QOX operator can produce nine potential offspring. Each algorithm is run 20 times independently for each numerical example. All the three algorithms stop after 1000 generations.

5.3 Comparisons of Our Algorithm with MOEA/D and NSGA-II

In this subsection, the comparisons of our algorithm with MOEA/D and NSGA-II algorithms will be given through some numerical examples based on the data of the Shanghai Stock Exchange Market. Because the diversity and convergence of solutions are very important for evaluating the performance of the algorithms, in order to compare the performance of the different algorithms quantitatively, performance metrics are needed. In this paper, the following performance metrics are used to compare the performance of the different algorithms quantitatively: set coverage (C-metric) [32] and hypervolume indicator (HV) [33]. C-metric is defined as the percentage of the solutions in B that are dominated by at least one solution in A, i.e., $\mathrm{C}(A,B)=\frac{\left|\{u\in B \mid \exists v\in A: v \text{ dominates } u\}\right|}{|B|}$, $\mathrm{C}(A,B)$ is not necessarily equal to $1-\mathrm{C}(A,B)$. $\mathrm{C}(A,B)=1$ means that all solutions in B are dominated by some solutions in A, while $\mathrm{C}(A,B)=0$ implies that no solution in B is dominated by a solution in A. The hypervolume indicator has been used widely in evolutionary multi-objective optimization to evaluate the performance of search algorithms. It computes the volume of the dominated portion of the objective space relative to a reference point. Higher values of this performance indicator imply more desirable solutions. The hypervolume indicator measures both the convergence to the true Pareto front and diversity of the obtained solutions.

Furthermore, in order to show the usefulness of our proposed model more clearly, We consider the following three cases: (I) returns of all assets remain unchanged, (II) returns of all assets become more increasing up to 10 percent than the sample returns in Table 1, (III) returns of all assets become more decreasing up to 10 percent than the sample returns in Table 1.

Table 3 presents the means of C-metric value obtained by our algorithm, MOEA/D, and NSGA-II with two models, respectively. It clearly shows that all of the solutions obtained by our algorithm dominate the solutions obtained by the NSGA-II and MOEA/D.

Table 3. Average set coverage between our algorithm (A), MOEA/D (B) and NSGAII (E)

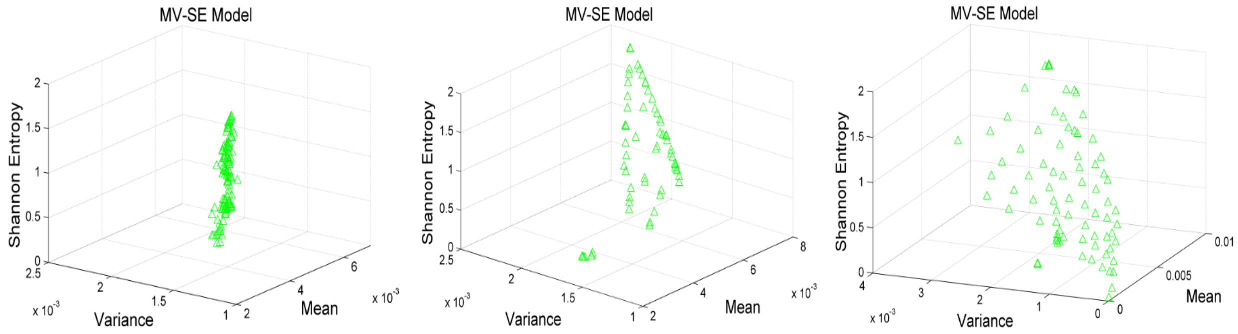
C-metric	C(A,B)	C(B,A)	C(A,E)	C(E,A)
Case (I)				
MV-LPE Model	0.0759	0	0.1457	0
MV-SE Model	0.8349	0	0.8933	0
Case (II)				
MV-LPE Model	0.8737	0	0.9994	0
MV-SE Model	0.3800	0	0.4251	0
Case (III)				
MV-LPE Model	0.8629	0	0.9990	0
MV-SE Model	0.8857	0	0.9473	0

Table 4 shows the mean and standard deviation of HV acquired by our algorithm, MOEA/D and NSG-AII, respectively. Best solutions acquired are highlighted bold in this table. According to the mean value of HV, the values of HV acquired by our algorithm are much bigger than those acquired by other two algorithms, which means that the diversities of solutions obtained by our algorithm are better than those obtained by MOEA/D and NSGA-II.

Table 4. HV obtained by our algorithm, MOEA/D and NSGAII

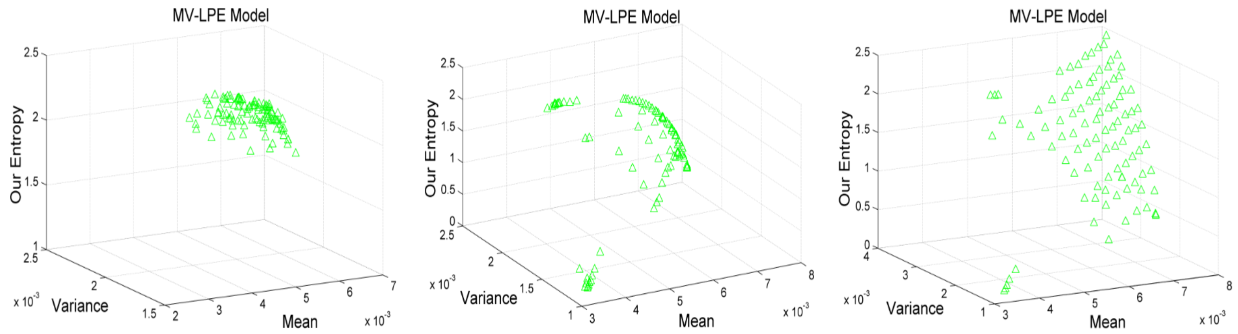
HV	MV-LPE Model Mean	SD	MV-SE Model Mean	SD
Case (I)				
Our Algorithm	**0.1559e-005**	**0.1346e-007**	**0.2710e-005**	**0.4451e-006**
MOEA/D	0.1565e-005	0.5184e-007	0.1548e-005	0.4817e-008
NSGA-II	0.1365e-005	0.4348e-007	0.1376e-005	0.4348e-007
Case (II)				
Our Algorithm	**0.1786e-004**	**0.1455e-008**	**0.0689e -005**	**0.1121e-006**
MOEA/D	0.1547e-005	0.1715e-007	0.0583e-005	0.0443e-007
NSGA-II	0.1533e-005	0.3516e-006	0.0503e-005	0.0261e-005
Case (III)				
Our Algorithm	**0.1763e-004**	**0.0925e-007**	**0.9712e-005**	**0.2196e-006**
MOEA/D	0.1588e-005	0.5615e-007	0.7994e-005	0.9279e-007
NSGA-II	0.1469e-005	0.4625e-006	0.7151e-005	0.0043e-006

In order to visually compare the performance of the three algorithms, the obtained non-dominated solutions by them on MV-SE and MV-PLE models are shown in Fig. 1 and Fig. 2, respectively.



(a) NSGA-II (b) MOEA/D (c) Our algorithm

Fig. 1. Comparisons of Pareto fronts obtained by three compared algorithms in MV-SE model



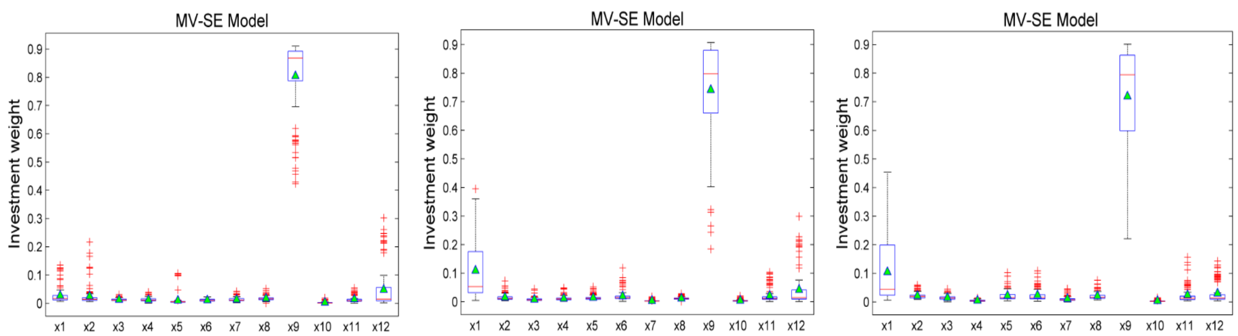
(a) NSGA-II (b) MOEA/D (c) Our algorithm

Fig. 2. Comparisons of Pareto fronts obtained by three compared algorithms in MV-LPE model

Obviously, according to Fig. 1 and Fig. 2, we can find that both NSGA-II and MOEA/D cannot locate the uniformly distributed non-dominated solutions comparing with our algorithm. In contrast, our algorithm can obtain a diverse population of non-dominated solutions. These Figures indicate that, in both the aspects of uniformity and diversity of solutions, our algorithm gives better performance than NSGA-II and MOEA/D.

5.4 Comparisons of Model MV-LPE, MV-SE and P-MV

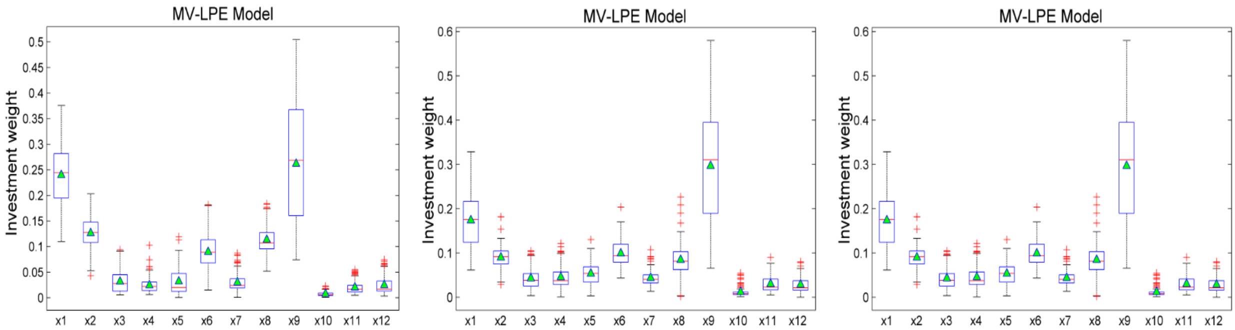
Fig. 3 and Fig. 4 show the box plots of average portfolio allocations that correspond to the portfolio models (MV-SE and MV-LPE) by using our designed algorithm, where x_i is the percentage of the sharing fund corresponding to the ith asset. In the Fig. 4, the symbol "Δ" represents the mean of x_i. The box in the plot contains 50% of the data points from the 25th to 75th percentile, and the red line drawn across the box is the median of x_i. The whiskers are lines extending above and below each box.



(a) MV-SE Model in Case (I) (b) MV-SE Model in Case (II) (c) MV-SE Model in Case (III)

Fig. 3. Box plots of average portfolio allocations for the MV-SE model in three cases

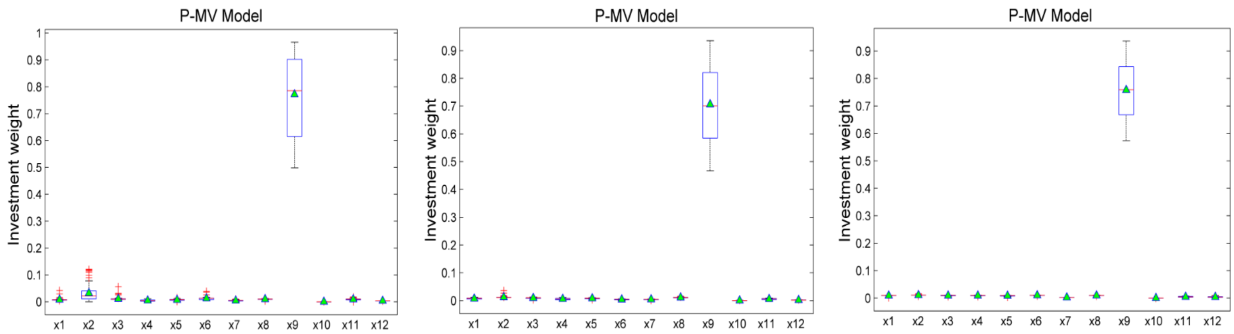
It can be seen from Fig. 3 that although the MV-SE model can obtain diversified portfolios, it yields very small weights for the other assets in the portfolios. The vast majority of investment weights for the MV-SE model extremely concentrate on the 9th asset, the portfolio weights of other assets are very small, even negligible, which is a contradiction to the notion of diversification. In essence, the MV-SE model generating the low diversity of the portfolios may result in loss while some of the invested assets will experience unexpected gains. This is because that the Shannon's entropy measure is non-linear in portfolio selection.



(a) MV-LPE Model in Case (I) (b) MV-LPE Model Case (II) (c) MV-LPE Model Case (II)

Fig. 4. Box plots of portfolio weights for the MV-LPE model in three cases

From the Fig. 4, we can see that the MV-LPE model can directly obtain more diversified portfolios because of the linear feature of our entropy. By comparing Fig. 3 with Fig. 4, it is clear to see that MV-LPE model can provide well-diversified and uniformed Pareto optimal solutions than the MV-SE model in the decision space. This means that MV-LPE model can allocate more diversified portfolio weights to all the twelve assets than the MV-SE model. Furthermore, it demonstrates that the performance of our proposed proportion entropy is better than the Shannon's entropy for measuring portfolio diversification.



(a) P-MV Model in Case (I) (b) P-MV Model Case (II) (c) P-MV Model Case (III)

Fig. 5. Box plots of average portfolio allocations for the conventional P-MV model in three cases

It can be seen from Fig. 5 that the portfolios extremely concentrate in the 9th asset, the portfolio weights of other assets are almost zero. Comparing Fig. 5 with Fig. 3 and Fig. 4, we can find that the conventional P-MV model will lead to extremely low diversity portfolios. Although, the MV-SE model performs the under-diversification compared with the MV-LPE model, it provides better diversity than the conventional P-MV model. In addition, according to the Fig. 3 to Fig. 5, we can find that including entropy in the mean-variance model can enhance diversity of the portfolios. The results of these experiments are consistent with the theoretical foundation which we mentioned earlier.

To compare the results of the portfolio models, we also present the Sharpe ratio of the models P-MV, MV-SE and MV-LPE. Table 5 shows the statistical results of Sharpe ratio for each model, including minimum, maximum, mean, and standard deviation. It can be seen that the statistical indicators of the Sharpe ratio of the MV-LPE model are higher than those of the P-MV and MV-SE model. This means that the performance of the MV-LPE model is higher than that of the P-MV and MV-SE model. This might attribute to the addition of function objective and our proposed linear proportion entropy enhances the mean-variance efficiency.

Table 5. Summary statistics of Sharpe ratio of portfolio model MV-SE and MV-LPE

Sharpe ratio	Min	Max	Mean	SD
Case (I)				
P-MV Model	0.0305	0.1602	0.1076	0.0318
MV-SE Model	0.0454	0.1610	0.1229	0.0294
MV-LPE Model	0.0829	0.1616	0.1376	0.0189
Case (II)				
P-MV Model	0.0312	0.1765	0.1099	0.0264
MV-SE Model	0.0703	0.1824	0.1429	0.0290
MV-LPE Model	0.0869	0.1836	0.1579	0.0207
Case (III)				
P-MV Model	0.0273	0.1389	0.0876	0.0317
MV-SE Model	0.0387	0.1396	0.1139	0.0286
MV-LPE Model	0.0624	0.1507	0.1169	0.0176

In the real investment circumstances, investors pay more attention to the profitability of executing these asset strategies. Furthermore, we discuss their economic benefits in portfolio management. Table 6 shows the comparison of real total return for each model in three cases, including minimum, maximum, mean, and standard deviation. We can see that the statistical indicators of the total return of the MV-LPE model are larger than those of the corresponding P-MV's and MV-SE's, in addition to the standard deviation. The smaller standard deviation of the total return means that the MV-LPE model can provide more stable investment strategies with large total return. It means that the MV-LPE model is more efficient than the P-MV and MV-SE models, which is consistent with the conclusions of the Table 5.

Table 6. Summary statistics of total return of the MV-LPE and MV-SE model

Total return	Min	Max	Mean	SD
Case (I)				
P-MV Model	1.8442e-004	0.0076	0.0041	0.0022
MV-SE Model	2.2184e-004	0.0078	0.0046;	0.0019
MV-LPE Model	0.0031	0.0078	0.0061	0.0013
Case (II)				
P-MV Model				
MV-SE Model	2.6460e-004	0.0088	0.0055	0.0021
MV-LPE Model	0.0036	0.0089	0.0070	0.0014
Case (III)				
P-MV Model	2.9068e-004	0.0066	0.0040	0.0017
MV-SE Model	1.9263e-004	0.0068	0.0044	0.0019
MV-LPE Model	0.0026	0.0082	0.0052	0.0012

The results of numerical examples suggest that our proposed algorithm can significantly outperform MOEA/D and NSGA-II. In addition, we find that the MV-LPE model can provide well-diversified Pareto optimal solutions than the P-MV and MV-SE models in the decision space, and the MV-LPE model can make asset allocation more feasible than the P-MV and MV-SE models. Our results also show that the performance of our proposed proportion entropy is better than the Shannon's entropy for measuring portfolio diversification.

6 Conclusions

One of the basic principles of financial investment is diversification where investors should allocate their capital among different assets. But, the distinguished drawbacks of the conventional mean-variance portfolio model are to generate extreme portfolio weights or low diversity in the portfolio. Although, many researchers proposed to use the Shannon's entropy function to measure the diversification in their portfolio selection models. However, the major issue of these portfolio models is that the Shannon's entropy measure is non-linear in portfolio selection since its weights take natural logarithm, which can result in very small positive weights for some asset in the portfolios.

To overcome these defects, in this paper, we propose a new proportion entropy function as an objective function to generate a well-diversified portfolio. Furthermore, in order to achieve the better portfolio selection and set up a reasonable portfolio selection model, by considering the five criteria: return, risk, transaction cost, liquidity and diversification degree of portfolio, we present a new fuzzy mean-variance-entropy multi-objective portfolio selection model to find tradeoffs between risk, return and the diversification degree of portfolio, which is able to address a more realistic portfolio selection problem. In addition, we design a multi-objective evolutionary algorithm based on objective space decomposition to solve our presented multi-objective portfolio selection model, which can maintain the diversity of obtained solutions. Finally, to demonstrate the efficiency and effectiveness of the proposed model and algorithm, we compare the performance of our designed algorithm with classic MOEA/D and NSGA-II algorithms through some numerical examples based on the data of the Shanghai Stock Exchange Market. Simulation results show that our designed algorithm is able to obtain better diversity and more evenly distributed Pareto fronts than the other two algorithms, and our proposed portfolio model can yield good performance of portfolio.

For the future research, the multi-objective fuzzy portfolio selection problem and MOEAs will be applied to other asset allocation problems, mutual fund portfolio selection problems, combinational optimization models and multi-period problems. Asset returns can be considered as random fuzzy variables. Also the proposed model can be applied for other case studies. Moreover, the new proposed models of portfolio selection problems and their efficient solution methods will help us to solve more complicated problems in real situations under more imprecise and ambiguous conditions.

Acknowledgments

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