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A New Filled Function Method with Two Parameters for Global Optimization

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Abstract. The filled function method is an effective approach to find the global minimizer of multi-modal functions because of its strict theoretical framework. Most of the conventional filled functions are numerical unstable due to exponential or logarithmic term and sensitive to parameters; in addition, most filled functions are discontinuous and non-differentiable that might influence effectiveness of the algorithm. In this paper, a new filled function with two parameters is proposed for solving global optimization problem, and several important theorems are proved. Unusually, this proposed filled function is continuously differentiable and non-sensitive to all parameters; although this new filled function contains two parameters, all of the parameters are easily set in numerical experiments. Based on these, a new filled function algorithm is proposed and tested on several benchmark functions. The numerical experiment results show that the new filled function algorithm is efficient. In addition, the proposed algorithm is compared, and the results indicate that the proposed filled function method can find more optimal solutions.

Keywords: filled function, global optimization, local minimizer, global minimizer

1 Introduction

More and more practical problems in science, economics, engineering and other fields can be formulated as global optimization problems. Lots of researchers have been attracted to the field of global optimization. In recent years, many new theoretical and computational contributions have been reported for solving global optimization problems. Global optimization is mainly concerned with the characteristics and algorithms on the multi-modal functions. In general, the existing approaches can be classified into two categories: deterministic methods [1-30] and probabilistic methods [31-36]. The typical examples of the former are the filled function method (FFM) [1-25], the trajectory method [26-27], the tunneling method [28], and the covering method [29, 30], whereas ones of the latter are the clustering method [31] and the methods reported in [32-33], the simulated annealing method [34] and genetic algorithms [35-36].

However, the existence of multiple local minima of a general nonconvex objective function makes global optimization become a great challenge. For global optimization problems, there are two major issues:

(1) How to find a lower minimizer of the objective function from a known local minimizer.

(2) How to evaluate the convergence and, accordingly, design the stopping criteria.

In this paper, we shall focus our research on the FFM, and mainly issue (1). Among the existing methods for global optimization problems, the FFM appears to have several advantages over others mainly due to its relatively easy realization with a process that aims at successively finding smaller local minima. The FFM was firstly proposed by Ge in [1], which was used to solve the global minimizer of unconstrained

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multi-extremum function. Later, many scholars have also done a lot of valuable works to improve this method [3-10, 37-41]. However, conventional filled functions are often nondifferentiable [8], need more than one adjustable parameters [9-10], or contain ill-conditioned terms [1-2]. To overcome these shortcomings, some parameter-free filled functions [38-39] and some filled functions without ill-conditioned terms [40] are proposed, however, they are usually nondifferentiable, which often results in additional local minimizers. And a continuously differentiable filled function with one parameter has been proposed [14], but the parameter is not easy to adjust. To deal with this problem, a new class of filled function with two parameters, which is continuously differentiable and all of the parameters is easy to adjust, is proposed in this paper. Based on this, a new filled function method is proposed, and numerical experiment shows that the methods are efficient and numerical stability. In addition, the proposed method can be used to solve the multidimensional problem.

2 Overview of the FFM

In this paper, we consider the following global optimization problem:

$$(P) \qquad \begin{cases} \min \ f(x) \\ s.t. \quad x \in R^n. \end{cases}$$

where $f(x): R^n \to R$ is a twice continuously differentiable function. Suppose $f(x)$ satisfies the condition $f(x) \to +\infty$ as $\|x\| \to +\infty$. Then there exists a closed bounded domain Ω called operating region that contains all local minimizers of $f(x)$. Then the global optimization problem (P) can be rewritten into an equivalent form as follows.

$$(P1) \qquad \begin{cases} \min \ f(x) \\ s.t. \quad x \in \Omega = [l, u] = \{x | l \le x \le u, l, u \in R^n\}. \end{cases}$$

Because Ω can be estimated before problem (P) is solved, so we can assume that Ω is known without loss of generality. We only consider problem (P1) in the following.

2.1 Basic Concepts and Assumptions

In this paper, we adopt the following symbols.

k : The iteration number;

$x_k{'}$: The initial point in the *k*-th iteration;

x_k^* : The local minimizer of the objective function in the *k*-th iteration;

f_k^* : The function value at x_k^*;

B_k^* : The basin of $f(x)$ at an isolated local minimizer x_k^*;

x^* : The global minimizer of the objective function.

Assumption 1. The function $f(x)$ in (P1) is continuously differentiable in R^n and $f(x)$ has only a finite number of minimizers in Ω, and therefore every minimizer is an isolated minimizer.

The basin B_k^* of $f(x)$ at a local minimizer x_k^* is defined in [1, 2] as a connected domain, and it contains x_k^* and the steepest descent trajectory of $f(x)$ converges to x_k^* from any initial point in B_k^*. The minimal radius of B_k^* at an isolated minimizer x_k^* is defined as

$$\overline{R} = \inf_{x \notin B_k^*} \|x - x_k^*\|. \tag{1}$$

Radius $\overline{R}$ is not zero if $\nabla^2 f(x_k^*)$ is positive definite. The basin of $f(x)$ at x_k^* is said to be lower than another basin of $f(x)$ at another local minimizer x_1^* if and only if $f(x_k^*) < f(x_1^*)$. The hill of $f(x)$ at x_k^* is the basin of $-f(x)$ at its isolated minimizer x_k^*.

The definition of the filled function was first proposed by Ge in [1] as follows.

Definition 1. Suppose x_k^* is a current local minimizer of $f(x)$. $P(x,x_k^*)$ is said to be a filled function of $f(x)$ at x_k^*, if it satisfies the following properties:

(i) x_k^* is a strictly maximizer of $P(x,x_k^*)$ and the whole basin B_k^* of $f(x)$ at x_k^* becomes a part of a hill of $P(x,x_k^*)$;

(ii) $P(x,x_k^*)$ has no minimizers or stable points in any basin of $f(x)$ higher than B_k^*;

(iii) If $f(x)$ has a lower basin than B_k^*, then there is a point x_{k+1}' in such a basin that minimizes $P(x,x_k^*)$ on the line through x and x_k^*.

Based on the filled functions, a global optimization problem can be solved via a two-phase cycle.

In Phase 1, we start from an initial point and use any local minimization method to find a local minimizer x_k^* of $f(x)$.

In Phase 2, we construct a filled function at x_k^* and minimize the filled function in order to identify a point x_{k+1}' with $f(x_{k+1}') < f(x^*)$. If such a point x_{k+1}' is found, x_{k+1}' is certainly in a lower basin than B_k^*. We can then use x_{k+1}' as an initial point in Phase 1 again, and hence we can find a better local minimizer x_{k+1}^* of $f(x)$ with $f(x_{k+1}^*) < f(x_k^*)$. This process repeats until no better solution can be found. The final local minimum will then be taken as a global minimizer of $f(x)$.

2.2 Overview of the FFM

As a deterministic yet universal global optimization technique, the development of the FFM undergoes the following generations. The representative examples of the FFM in the first generation are *P*-functions [1] and *G*-functions [2] given by

$$P(x,r,\rho) = \exp\left(-\left\|x - x_k^*\right\| / \rho^2\right) / \left[r + f(x)\right], \tag{2}$$

$$G(x,r,\rho) = -\left\{\rho^2 \ln\left[r + f(x)\right] + \left\|x - x_k^*\right\|^p\right\}. \tag{3}$$

The first-generation filled functions share a common feature: there are two adjustable parameters, r and ρ, which greatly affect the performance of the algorithms and need to be appropriately adjusted. However, how to adjust the parameters is a very difficult task. Due to these drawbacks, the second-generation filled functions were proposed and they have only one parameter. Among them, the best known one is perhaps the *Q*-function [2]:

$$Q(x,a) = -\left[f(x) - f(x_k^*)\right]\exp\left(a\left\|x - x_k^*\right\|^2\right). \tag{4}$$

where the adjustable parameter is *a*. This filled function is significantly simple than those in the first generation. However, the *Q*-function is liable to be ill-conditioned in practice since its function value increases exponentially due to an exponential function in it. As *a* becomes larger and larger, which is required by the FFM itself, the rapidly increasing exponential function value may result in an overflow in the computation. To overcome this drawback, the *H*-function was proposed by Liu [40]:

$$H(x,a) = \frac{1}{\ln\left[1 + f(x) - f(x_k^*)\right]} - a\left\|x - x_k^*\right\|^2 \tag{5}$$

The *H*-function retains the advantage of the *Q*-function with only a single parameter and without exponential terms. The performance of the *H*-function in numerical experiments for a large set of testing functions was quite satisfactory [40]. *H*(*X,a*) can be regarded as the third generation due to the absence of the exponential term. Nevertheless, the *H*-function has a drawback which is discontinuous at $x \in S = \left\{x / f(x) = f(x_k^*)\right\}$.

However, in one hand, the continuity and differentiability on the FFM are required to the convergence analysis in theory [10]. On the other hand, computationally, most local minimization algorithms for the numerical nonlinear programming require the gradients information in their procedures (readers can refer to [42] or [43] for detail). Thus, it is very necessary to develop a continuously differentiable filled function with as few parameters. There are already some works in this area [e.g., 14], but the parameters of the filled functions are not easy to adjust. Based on this consideration, a continuously differentiable filled function with two parameters is designed, and the parameters is relatively easy to adjust and insensitive.

3 A New Filled Function and Its Properties

Definition 1 relies on the concept of the basin and hill of $f(x)$, which requires that the minima in the operating region are isolated, and Definition 1 also requires there exists a minimal point of $f(x)$ along a line. This is more difficult to be guaranteed. Therefore, many improvements in the definition are given in the literatures [e.g., 41], which make it more convenient to construct a new filled function. In this section, we use the revised definition in [41] for the problem (P1).

Definition 2. Suppose x_k^* is a current local minimizer of $f(x)$. $P(x,x_k^*)$ is said to be a filled function of $f(x)$ at x_k^*, if it satisfies the following properties:

(1) x_k^* is a strictly maximizer of $P(x,x_k^*)$;

(1) For any $x\in\Omega_1$, one has $0\notin\partial P(x,x_k^*)$, where $\Omega_1=\left\{x\in\Omega / f(x)\ge f(x_k^*), x\ne x_k^*\right\}$;

(1) If $\Omega_2=\left\{x / f(x)<f(x_k^*), x\in\Omega\right\}$ is not empty, then there exists a point $x_k{}'\in\Omega_2$ such that $x_k{}'$ is a local minimizer of $P(x,x_k^*)$.

Note that Definition 2 about the filled function is different from the definition mentioned in [1]. It is much easier to construct a new filled function by Definition 2 and the local optimal solution of the filled function can be easily found. For example, suppose that x_k^* is not a global minimizer, then by condition (III) of definition 2, we can find a point $x_k{}'\in\Omega_2$ by minimizing $P(x,x_k^*)$. Therefore, we can obtain a local minimizer x_{k+1}^* of $f(x)$ by searching $f(x)$ starting at $x_k{}'$ via local search schemes. In the process of minimizing $P(x,x_k^*)$, it does not require, unlike the definition 1, that x_{k+1}^* must be on the straight line through $x_k{}'$ and x_k^*. So the design of the filled function is much easier and flexible.

In order to find a global minimizer of $f(x)$, the major issue of the filled function method is to find a lower minimizer of $f(x)$ or justify whether the obtained local minimizer is a global minimizer of $f(x)$. This heavily relies on the performance of the filled function used.

In this section, we propose a new filled function for problem (P1) at a local minimizer x_k^* as follows:

$$P(x,x_k^*)=-\left\|x-x_k^*\right\|^2+g\left(f(x)-f(x_k^*)\right),\qquad g(t)=\begin{cases}0, & t\ge 0,\\ r\cdot\arctan\left(t^{\rho}\right), & t<0.\end{cases}, \tag{6}$$

where r is an adjustable positive real number as large as possible, used as the weight factor, and $\rho>0$ is an even number.

Note that the proposed filled function has some advantages: first, the parameter r is a positive real number as large as possible, thus it is easy to adjust, and second, it is continuously differentiable, which makes it more easily solved by the existing local optimization method, and finally, $\arctan\left(t^{\rho}\right)\in\left[0,\frac{\pi}{2}\right)$ is bounded, which ensures that the calculation of $P(x,x_k^*)$ will not overflow and is numerical stability. The following theorems show that $P(x,x_k^*)$ satisfies Definition 2.

Theorem 1. Suppose that x_k^* is a local minimizer of $f(x)$, then x_k^* is a strictly local maximizer of $P(x,x_k^*)$.

Proof. Since x_k^* is a local minimizer of $f(x)$, then there exists a small positive real number ε, and a neighborhood $\delta = U(x_k^*,\varepsilon)$, such that for all $x \in \delta$, $x \neq x_k^*$, and $f(x) > f(x_k^*)$.

Then $P(x,x_k^*) = -\|x - x_k^*\|^2 < 0 = P(x_k^*,x_k^*)$. Thus, x_k^* is a strictly local maximizer of $P(x,x_k^*)$.

Theorem 1 clearly shows that $P(x,x_k^*)$ satisfies the property (I) of Definition 2.

Theorem 2. Suppose Assumption 1 is satisfied, x_k^* is a local minimizer of $f(x)$, for any

$$x \in \Omega_1 = \left\{x / f(x) \geq f(x_k^*), x \in \Omega, x \neq x_k^*\right\}, \text{ one has } 0 \notin \nabla P(x,x_k^*).$$

Proof. For any $x \in \Omega_1$, $f(x) \geq f(x_k^*)$, and $x \neq x_k^*$, one has $\nabla P(x,x_k^*) = -2\left(x - x_k^*\right) \neq 0$. Consequently, $0 \notin \nabla P(x,x_k^*)$.

Theorem 3. Suppose Assumption 1 is satisfied, x_k^* is a local minimizer of $f(x)$, and $\Omega_2 = \left\{x / f(x) < f(x_k^*), x \in \Omega\right\}$ is not empty, then there exists a point $x_k' \in \Omega_2$ such that x_k' is a local minimizer of $P(x,x_k^*)$.

Proof.

(1) For any $x \in \Omega_2$, $P(x,x_k^*) = -\|x - x_k^*\|^2 + r \cdot \ln(\arctan\left(f(x) - f(x_k^*)\right)^\rho)$, $P(x,x_k^*) > 0$ is very easy to guarantee, when r is a positive real number as large as possible. Thus, there exists $r > 0$, and $\bar{x} \in \Omega_2$, such that $P(\bar{x},x_k^*) > 0$.

(2) For any $x \in \Omega_1$, $P(x,x_k^*) < 0$, and in theorem 2, for any $x \in \Omega_1$, one has $0 \notin \nabla P(x,x_k^*)$.

(3) In addition, $P(x,x_k^*)$ is continuously differentiable,

Thus, there exists a point $x_k' \in \Omega_2$, such that $\nabla P(x_k',x_k^*) = 0$, and x_k' is a local minimizer of $P(x,x_k^*)$.

Theorems 1-3 state that the proposed filled function satisfies properties of Definition 2.

4 Filled Function Algorithm

4.1 A Local Search Method

Conventional local optimization method minimize function $f(x)$ directly from the initial point, and then a local minimizer x_k^* is obtained. In the process, the function is called repeatedly when the derivative and the search direction are calculated, in addition, the number of function evaluations is also increased in the one-dimensional search process. More importantly, with the increase of dimension, the number of iterations is also increasing, and the number of function evaluations has also increased. Thus, the amount of computation for the traditional local search methods is very large, which would affect the computational efficiency. In this subsection, a local search strategy called randomly and uniformly local search (RULS) is given [11], which can make the initial point closer to the local optimal solution, thereby the number of iterations and the number of function evaluations will be reduced, and the convergence speed will be accelerated.

The details for RULS are as follows:

Step 1. $x_k' \in \Omega = [l,u] = \left\{x | l \leq x \leq u, l,u \in R^n\right\}$ is an initial point for the *k*-th iteration, and M is the set of all points that have been used so far. Calculate $a = \frac{\sum_{i=1}^{|M|-1} \|x_k' - x_i\|}{|M|-1}$, $x_i \in M$.

Step 2. $\Gamma \in \Omega$ is a neighborhood at x_k', where $\Gamma = [\Gamma_l, \Gamma_u] = \left[x_k' - \frac{a}{m}e, x_k' + \frac{a}{m}e \right]$, $m > 1$, and $e = (1, \cdots, 1)^T \in R^n$.

Randomly and uniformly select $b*n$ points $Y_k = \{y_1, y_2, \cdots, y_{bn}\}$ in Γ, where n is the dimension of the problem, and b is the number of points selected in each dimension, $M = \{M, Y_k\}$.

Step 3. Calculate the function value of these points $Y_k = \{y_1, y_2, \cdots, y_{bn}\}$ to find the points $\bar{Y}$ corresponding to the smallest function value, where

$$\bar{Y} = \underset{y_i \in \Gamma, i \in \{1,2,\cdots,bn\}}{\arg\min} \{f(y_1), f(y_2), \cdots, f(y_{bn})\}.$$

Let $X_1 = \underset{x_k, y \in \bar{Y} \cup x_k}{\arg\min} \{f(x_k), f(y)\}$ as a set of the initial points.

Step 4. Starting from any point $z_k \in X_1$, minimize $f(x)$ by using a local optimization method to obtain a local minimizer sequence $X_2 = \{z_k^*\}$ and a local minimum value sequence $F_2 = \{f(z_k^*)\}$, set $M = \{M, X_2\}$.

Step 5. Calculate $f(x_k^*) = \min\{f(z_k^*)\}$ to obtain a local minimizer x_k^*, $M = \{M, x_k^*\}$.

Explanation of RULS algorithm: uniformly and randomly generate some points near the initial point x_k', and select one of the points corresponding to the smallest function value as the new initial point to minimize the function $f(x)$, and obtain a local minimizer x_k^*.

4.2 Filled Function Algorithm

Based on the results of the previous content, a new filled function algorithm is proposed as follows.

Step 1. Initialization Step

(1) Choose a tolerance $\varepsilon > 0$, e.g. $\varepsilon := 1.0e-20$.

(2) Choose a large integer constant B and a positive real number as large as possible r, and a small constant $\delta := 1.0e-3$.

(3) Set $f(x_{k-1}^*) = +\infty$, and $k := 1$.

Step 2. Randomly and uniformly select $b*n$ points Y_k in the operating region, where $Y_k = \{y_1, y_2, \cdots, y_{bn}\}$, $y_i \in \Omega, i \in \{1, 2, \cdots, bn\}$, and n is the dimension of the problem, and b is the number of points selected in each dimension. Set $M = \{M, Y_k\}$, calculate the function value of these points, and find the points corresponding to the smallest function value $\bar{Y} = \underset{y_i \in \Gamma, i \in \{1,2,\cdots,bn\}}{\arg\min} \{f(y_1), f(y_2), \cdots, f(y_{bn})\}$. Select $\forall x_k \in \bar{Y}$.

Step 3. Starting from x_k, minimize $f(x)$ by using RULS to obtain a local minimizer x_k^*, set $M = \{M, x_k^*\}$ and go to Step 6.

Step 4. Construct

$$P(x, x_k^*) = -\|x - x_k^*\|^2 + g\left(f(x) - f(x_k^*)\right),$$

$$g(t) = \begin{cases} 0, & t \geq 0, \\ r \cdot \arctan(t^2), & t < 0. \end{cases}$$

Step 5. Set $x = x_k^* + \delta e^i$, use x as the initial point to minimize $P(x, x_k^*)$ by using RULS, and find minimizers x' of $P(x, x_k^*)$. Set $M = \{M, x'\}$, $x_{k+1} = x'$, and $k = k + 1$, and then go to step 3.

Step 6. Termination step

If $k < B$ or $\left| f(x_k^*) - f(x_{k-1}^*) \right| \le \varepsilon,\ k = 1,2,\cdots$, or $P(x, x_k^*)$ has no stable point, the algorithm stops and $x^* = x_k^*$ is taken as a global minimizer of $f(x)$; Otherwise, increase r and go to step 4.

Some explanations about the above filled function algorithm are necessary.

(1) In minimization to $f(x)$ and $P(x, x_k^*)$, a local optimization method is needed to select firstly. In our algorithm, the Matlab function 'fmincon' is used.

(2) In step 5, the smaller δ is needed to select accurately, in the algorithm, the δ is selected to guarantee that $\left\| \nabla P(x, x_k^*) \right\|$ is greater than a threshold.

(3) Step 5 means that if local minimizer x' of $P(x, x_k^*)$ is found in Ω and with $f(x') < f(x_1^*)$, we can use x' as the initial point to minimize $f(x)$ and obtain a better local minimizer of $f(x)$.

5 Numerical Experimentation

5.1 Test Problems

In this section, the proposed algorithm is tested on problem 1 and some benchmark problems 2-9 taken from [9].

Problem 1. (One-dimensional function)

$$\min f(x) = \sin x + \sin 2x - \cos 4x, \quad s.t.\ -2 \le x \le 4,$$

The global minimizer is $x^* = -1.4523$, and the global optimal value is $f(x^*) = -2.1175$.

Problem 2. (Two-dimensional function)

$$\min f(x) = \left[1 - 2x_2 + c\sin\left(4\pi x_2\right) - x_1\right]^2 + \left[x_2 - 0.5\sin\left(2\pi x_1\right)\right]^2, \quad s.t.\ 0 \le x_1 \le 10,\ -10 \le x_2 \le 0,$$

where $c = 0.2, 0.5, 0.05$. The global minimum value is $f(x^*) = 0$ for all c.

Problem 3. (Three-hump back camel function)

$$\min f(x) = 2x_1^2 - 1.05x_1^4 + \frac{1}{6}x_1^6 - x_1x_2 + x_2^2, \quad s.t.\ -3 \le x_1 \le 3,\ -3 \le x_2 \le 3,$$

The global minimizer is $x^* = \left(0,0\right)^T$, and the global optimal value is $f(x^*) = 0$.

Problem 4. (Six-hump back camel function)

$$\min f(x) = 4x_1^2 - 2.1x_1^4 + \frac{1}{3}x_1^6 - x_1x_2 - 4x_2^2 + 4x_2^4, \quad s.t.\ -3 \le x_1 \le 3,\ -3 \le x_2 \le 3,$$

The global minimizers are $x^* = \left(-0.0898, -0.7127\right)^T$ and $x^* = \left(0.0898, 0.7127\right)^T$, and the global optimal value is $f(x^*) = -1.0316$.

Problem 5. (Treccani function)

$$\min f(x) = x_1^4 + 4x_1^3 + 4x_1^2 + x_2^2, \quad s.t.\ -3 \le x_1 \le 3,\ -3 \le x_2 \le 3,$$

This problem has two global minimizers in total, which are $x^* = \left(0,0\right)^T$ and $x^* = \left(-2,0\right)^T$, and the global optimal value is $f(x^*) = 0$.

Problem 6. (Goldstein and Price function)

$$\min f(x) = g(x)h(x),$$
$$s.t.\ -3 \le x_1 \le 3,\ -3 \le x_2 \le 3,$$

Where $g(x) = 1 + (x_1 + x_2 + 1)^2 (19 - 14x_1 + 3x_1^2 - 14x_2 + 6x_1x_2 + 3x_2^2)$, and

$$h(x) = 30 + (2x_1 - 3x_2)^2 (18 + 32x_1 + 12x_1^2 + 48x_2 - 36x_1x_2 + 27x_2^2).$$

The global minimizer is $x^* = (-2.6852, -3.0000)^T$, and the global optimal value is $f(x^*) = -9.6233e + 006$.

Problem 7. (Two-dimensional Shubert function)

$$\min f(x) = \left\{ \sum_{i=1}^{5} i \cos\left[(i+1)x_1 + i\right] \right\} \left\{ \sum_{i=1}^{5} i \cos\left[(i+1)x_2 + i\right] \right\},$$
$$s.t.\ 0 \le x_1 \le 10,\ 0 \le x_2 \le 10,$$

This problem has two global minimizers in total, which are $x^* = (4.8581, 5.4829)^T$ and $x^* = (5.4829, 4.8581)^T$. The global minimum value is $f(x^*) = -186.7309$.

Problem 8. (Shekel's function)

$$\min f(x) = -\sum_{i=1}^{5} \left[\sum_{j=1}^{4} (x_j - a_{i,j})^2 + c_i \right]^{-1},$$
$$s.t.\ 0 \le x_j \le 10, j = 1,2,3,4,$$

where the coefficients $a_{i,j}, c_i$, $i = 1,2,3,4$, $j = 1,2,3,4$ are given in the following:

Table 1. $a_{i,j}$, c_i, $i = 1,2,3,4$, $j = 1,2,3,4$

i	$a_{i,1}$	$a_{i,2}$	$a_{i,3}$	$a_{i,4}$	c_i
1	4.0	4.0	4.0	4.0	0.1
2	1.0	1.0	1.0	1.0	0.2
3	8.0	8.0	8.0	8.0	0.3
4	6.0	6.0	6.0	6.0	0.4
5	3.0	7.0	3.0	7.0	0.5

The global minimizer is $x^* = (4.0000, 4.0001, 4.0000, 4.0001)^T$ and the global optimal value is $f(x^*) = -10.1529$.

Problem 9. (n-dimensional function)

$$\min f(x) = \frac{\pi}{n}\left[10\sin^2 \pi x_1 + g(x) + (x_n - 1)^2\right],$$
$$s.t.\ -10 \le x_i \le 10, i = 1,2,\cdots,n,$$

where $g(x) = \sum_{i=1}^{n-1}\left[(x_i - 1)^2 (1 + 10\sin^2 \pi x_{i+1})\right]$. The global minimizer of this problem is $x^* = (1,\cdots,1)^T$ for all n.

In the following, the solving process schematics of test problem 1 are given respectively by using the conventional filled function method and the proposed algorithm, and comparison of these two algorithms is also given.

5.2.1 Conventional Filled Function Method for Solving the Test Problem 1

The calculation process are as follows:

(1) Randomly selected point x_1=1.043 in the operating region Ω, starting from x_1, minimize $f(x)$ by using local minimization function 'fmincon' to obtain a local minimizer $x_1^* = 1.71$.

(2) Construct filled function $P(x,x_1^*) = -\|x - x_1^*\|^2 + g\left(f(x) - f(x_1^*)\right)$ at $x_1^* = 1.71$, where $f(x) = \sin x + \sin 2x - \cos 4x$,

$$g(t) = \begin{cases} 0, & t \ge 0, \\ 4 \cdot \arctan\left(t^2\right), & t < 0. \end{cases}$$

(3) Minimize filled function $P(x,x_1^*)$ to obtain a local minimizer $x_2 = 0.18$ of $P(x,x_1^*)$; The specific process of Step1, 2, and 3 are shown in Fig. 1.

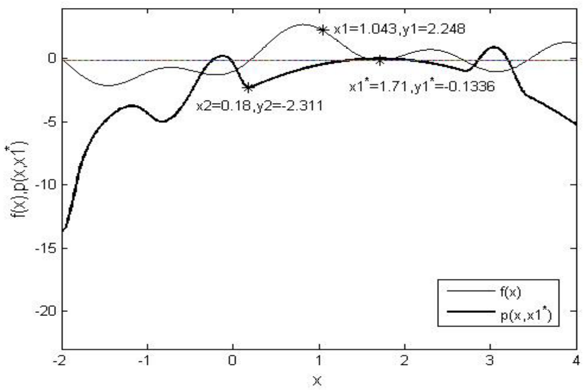

Fig. 1. The specific process of Steps 1, 2, and 3

(4) Starting from $x_2 = 0.18$ to minimize $f(x)$, obtain a local minimizer $x_2^* = \text{-}0.2$ of $f(x)$. Construct the filled function $P(x,x_2^*)$ at x_2^*, where $r = 3$, $\rho = 2$, and then minimize the filled function $P(x,x_2^*)$ to obtain a local minimizer $x_3 = -1.1$ of $P(x,x_2^*)$. Shown in Fig. 2.

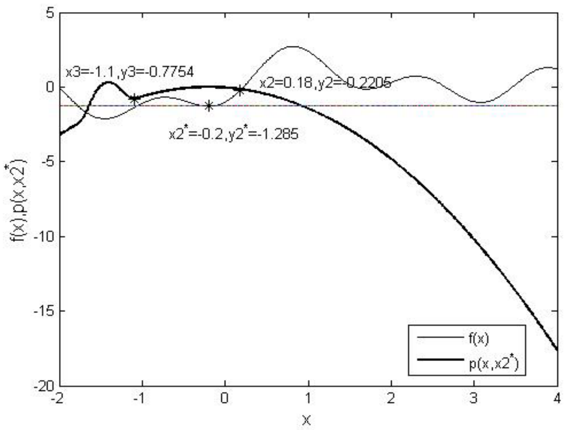

Fig. 2. The specific process of Step 4

(5) Starting from $x_3 = -1.1$ to minimize $f(x)$, a local minimizer $x_3^* = \text{-}1.4523$ is obtained. Construct the filled function $P(x,x_3^*)$ at x_3^*, where $r = 3$, $\rho = 2$, which is shown in Fig. 3, and then minimize $P(x,x_3^*)$, there is no any stable point is found. Then the iteration is terminated, set $x^* = x_3^*$, the global minimizer $x^* = \text{-}1.4523$ and the global minimum value $f(x^*) = -2.1175$ are obtained.

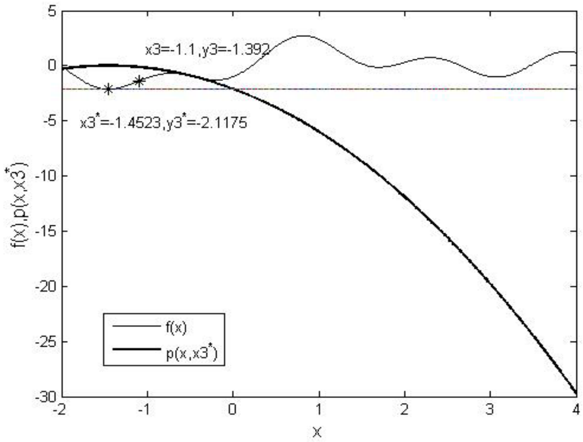

Fig. 3. The specific process of Step 5

Conventional filled function algorithm is run independently 50 times, the mean number of function evaluations is 19.5131.

5.2.2 The Proposed Filled Function Algorithm for Solving the Test Problem 1

The following use of the proposed filled function algorithm for solving the test problem 1, and the specific process are as follows:

(1) Uniform randomly generated three points $x_{01}=0,\ x_{02}=2.2,\ x_{03}=3.35$ in the operating region, set $M=\{x_{01},x_{02},x_{03}\}$, and calculate the function value $f(x)$ of the three points. Find the point $x_0=0$ corresponding to the smallest function value, as shown in Fig. 4.

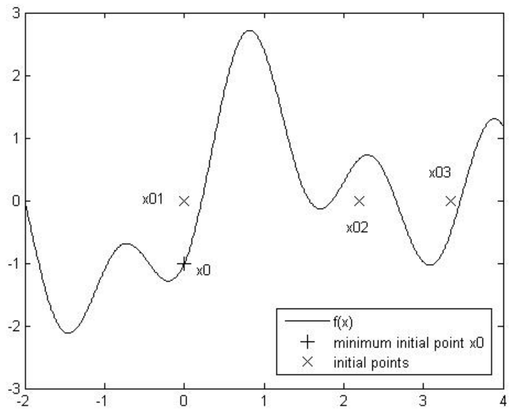

Fig. 4. The specific process of Step 1

(2) Calculated $a=\frac{\sum_{i=1}^{|M|-1}\|x_0-x_i\|}{|M|\text{-}1}=\frac{\|0-2.2\|+\|0-3.35\|}{2}=2.775$ by using the rules in RULS, where $x_i\in M$, let $m=5$, and then

$$\Gamma=\left[x_0-\frac{a}{m},x_0+\frac{a}{m}\right]=\left[x_0-0.555,x_0+0.555\right];$$

Randomly selected three points in Γ, and calculate their value of $f(x)$ to find the point x_1 corresponding to the smallest function value, as shown in Fig. 5, and set $M=\{M,x_1\}$.

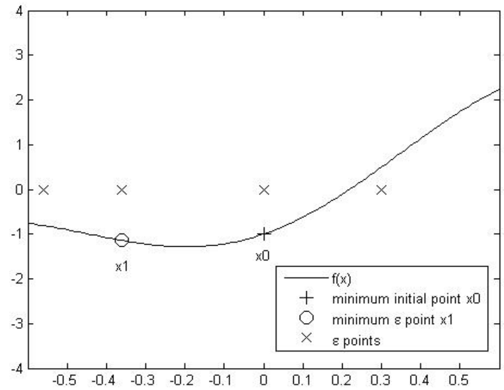

Fig. 5. The specific process of Step 2

(3) Starting from x_1, minimize $f(x)$ by using a local optimization method to find a local minimizer $x_1^* = -0.2$ and a minima $f(x_1^*) = -1.285$, as follows in Fig. 6, and set $M = \{M, x_1^*\}$;

Step 2, 3 describe the specific process of RULS, as shown in Fig. 5 and Fig. 6.

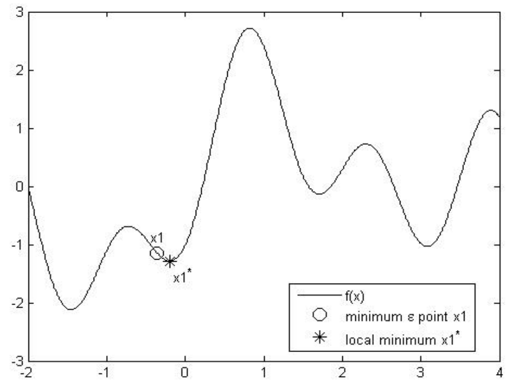

Fig. 6. The specific process of Step 3

(4) Construct $P(x, x_1^*) = -\left(x - x_1^*\right)^2 + g\left(f(x) - f(x_1^*)\right)$ at $x_1^* = -0.2$, as follows in Fig. 7, where $f(x) = \sin x + \sin 2x - \cos 4x$,

$$g(t) = \begin{cases} 0, & t \geq 0, \\ 3 \cdot \arctan\left(t^2\right), & t < 0. \end{cases}$$

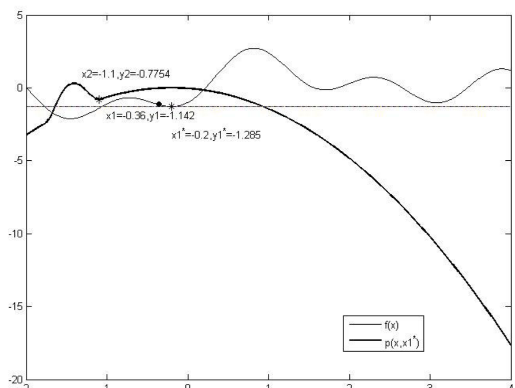

Fig. 7. The specific process of jumping x_1^*

(5) Using RULS method to optimize the function $P(x, x_1^*)$ at $x_1^* = -0.2$, obtain a minimizer $x_2 = -1.1$ of $P(x, x_1^*)$, as shown in Fig. 7, set $x_0' = x_2$, and $M = \{M, x_2\}$, and then minimize $f(x)$ by using RULS to obtain $x_2^* = -1.4523$, and $f(x_2^*) = -2.1175$;

(6) Construct a filled function $P(x, x_2^*)$ at x_2^*, where $r = 3$, $\rho = 2$, as follows in Fig. 8.

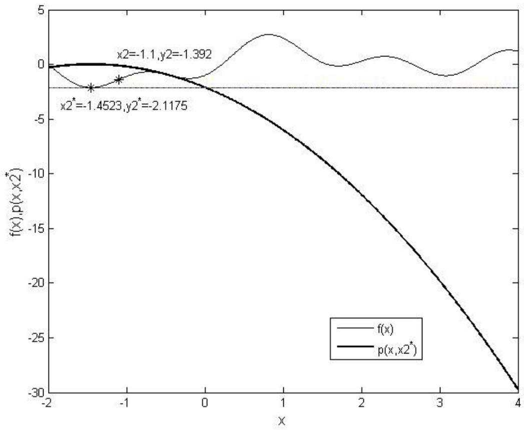

Fig. 8. The specific process of jumping x_2^*

(7) Function $P(x, x_2^*)$ has not a stable point in $\{x \in \Omega | x \neq x_2^*\}$, so the iterations terminate, set $x^* = x_2^*$; and the global minimizer $x^* = -1.4523$, and the global minimum value $f(x_2^*) = -2.1175$ are obtained.

The proposed algorithm is run independently 50 times, the mean number of function evaluations is 11.4358, which significantly less than the former result that of using conventional filled function method.

Both experimental results of 5.2.1 and 5.2.2 and the mean number of function evaluations show that the proposed filled function method is efficient.

5.2.3 The Proposed Filled Function Method for Solving the Benchmark Problems Taken from [9]

The proposed algorithm is executed on above 2-9 test problems. The results obtained by the proposed algorithm and the comparison with [9] are listed in Table 2 and Table 3.

The symbols used in Tables are given as follows:

No.: The order of the test problems;

n: The dimension of the test problems;

r: The parameter of the filled function;

ρ: The parameter of the filled function;

k: The iteration number;

x_k^*: The local minimizer of the objective function in the *k*-th iteration;

f_k^*: The function value at x_k^*;

x^*: The global minimizer of the objective function;

f-mean: The mean function value respectively in the 50 runs;

f-best: The best function value respectively in the 50 runs;

f-std: The standard deviation of function value respectively in the 50 runs;

Some explanation of the above experimental results:

The problems 2-9 are tested by using the proposed algorithm respectively in the 50 runs, and the detailed results are shown in Tables 2 and 3, and the specific numerical analysis and the comparison with [9] are reported as follows:

More minimizers could be obtained. In the proposed algorithm, the initial points are randomly uniformly generated, so that many different global minimizers are obtained respectively in the 50 runs. For example, Problems 2, 5, and 7 get more optimal solutions than in the literature [9].

Table 2. The results obtained by the proposed algorithms and the comparison with [9]

			The proposed algorithm		The algorithm of the literature [9]	
No.	r	ρ	x_k^*	f_k^*	x_k^*	f_k^*
2 (c=0.2) (n=2)	100	2	(1.5909; -0.2703) (0.9997; -0.0005) (1.8784; -0.3458) (0.9820; -0.0565)	0	(1.8784; -0.3458)	0
2 (c=0.5) (n=2)	100	2	(1;0) (1.5872; -0.2606) (1.8973; -0.3005)	0	(1.0000; -2.2205e-14)	0
2 (c=0.05) (n=2)	100	2	(1.8513; -0.4021) (1.5975; -0.2874)	0	(1.8513; -0.4020)	0
3 (n=2)	100	2	(0;0)	0	(0;0)	0
4 (n=2)	100	2	(0.0898; 0.7127) (-0.0898; -0.7127)	-1.0316	(-0.0898; -0.7126) (0.0898; 0.7126)	-1.0316
5 (n=2)	100	2	(-2;0) (0;0)	0	(0;0)	0
6 (n=2)	100	2	(-2.6852; -3.0000)	-9.62e+006	(0; -1.0000)	3.0000
7 (n=2)	1000	2	(4.8581; 5.4829) (5.4829; 4.8581)	-186.7309	(5.4829; 4.8581)	-186.73
8 (n=4)	100	2	(4.0000; 4.0001; 4.0000; 4.0001)	-10.1529	(4.0000; 4.0001; 4.0000; 4.0001)	-10.1529
9 (n=2)	100	2	(1;1)	0	(1;1)	0
9 (n=3)	100	2	(1;1;1)	0	(1;1;1)	0
9 (n=7)	1000	2	(1.0000; 1.0000; 1.0001; 1.0000; 1.0000; 1.0001; 1.0000)	0	(1.0000; 1.0000; 1.0000; 1.0000; 1.0000; 1.0000; 1.0000)	0
9 (n=10)	1000	2	(1.0000; 1.0000; 1.0000; 1.0000; 1.0000; 1.0000; 1.0000; 1.0000; 1.0000; 1.0000)	0	(1.0000; 1.0000; 1.0000; 1.0000; 1.0000; 1.0000; 1.0000; 1.0000; 1.0000; 1.0000)	0

Table 3. The results obtained by the proposed algorithms for solving the problem 9 with different dimensions

No.	r	ρ	*f-mean*	*f-best*	*f-std*
9 (n=12)	1.0e+6	2	1.2563e-006	1.0015e-010	1.0058e-008
9 (n=15)	1.0e+7	2	3.4013e-005	1.0601e-014	1.1012e-007
9 (n=20)	1.0e+9	2	1.1606e-005	6.2176e-009	6.0453e-007
9 (n=30)	1.0e+10	2	2.5012 e-003	5.5078e-007	3.4016e-005

Effectiveness of the algorithm. The minimizers and the optimal value can be found by using the proposed algorithm, which indicates the effectiveness of the algorithm. In particular, compared with [9], much smaller optimal value is obtained in problem 6 by using the proposed algorithm.

The stability of the algorithm. In Table 2, all the data in columns *f-mean* and *f-std* shows that the algorithm of this paper is stable.

The algorithm can be used to solve multidimensional problem. In Table 3, the problem 9 with different dimensions is tested, and the numerical results indicate that the proposed algorithm is suitable for solving multidimensional problem.

6 Conclusions

The filled function method is an approach to find the global minima of multi-modal functions. The existing filled functions have some drawbacks such as being non-differentiable at some point in search domain, including the exponential and logarithm terms, containing sensitive adjust parameters, and being discontinuous, and so on. In this paper, a filled function with two parameters is designed, which is continuously differentiable and insensitive to parameters, and it can overcome the former shortcomings in certain degree. Based on this, a new filled function method is proposed, and the algorithm is numerical stability. The comparison results of the computer simulations indicate that the proposed filled function method is effective and efficient.

Acknowledgements

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References

[1] R.P. Ge, A filled function method for finding a global minimizer of a function of several variables, Mathematical Programming 46(1990) 91-204.

[2] R.P. Ge, Y.F. Qin, A class of filled functions for finding global minimizers of a function of several variables, Journal of Optimization Theory and Applications 54(2)(1987) 241-252.

[3] A.F. Ling, C.X. Xu, F.M. Xu, A discrete filled function algorithm for approximate global solutions of max-cut problems, Journal of Computational and Applied Mathematics 220(1-2)(2008) 643-660.

[4] W.X. Wang, Y.L. Shang, L.S. Zhang, A filled function method with one parameter for box constrained global optimization,

Applied Mathematics and Computation 194(1)(2007) 54-66.

[5] S.X. He, W.L. Chen, H. Wang, A new filled function algorithm for constrained global optimization problems, Applied Mathematics and Computation 217(12)(2011) 5853-5859.

[6] H.W. Lin, Y.P. Wang, L. Fan, A filled function method with one parameter for unconstrained global optimization, Applied Mathematics and Computation 218(7)(2011) 3776-3785.

[7] X. Liu, Several filled functions with mitigators, Applied Mathematics and Computation 133(2-3)(2002) 375-387.

[8] S.Z. Ma, Y.J. Yang, H.G. Liu, A parameter free filled function for unconstrained global optimization, Applied Mathematics and Computation 215(10)(2010) 3610-3619.

[9] L.S. Zhang, C.K. Ng, D. Li, W.W. Tian, A new filled function method for global optimization, Journal of Global Optimization 28(1)(2004) 17-43.

[10] X. Liu, A class of continuously differentiable filled functions for global optimization. IEEE Transactions on Systems, Man, and Cybernetics- Part A: System and Humans 38(1)(2008) 38-47.

[11] F. Wei, Y.P. Wang, H.W. Lin, A new filled function method with two parameters for global optimization, Journal of Optimization Theory and Applications 163(2)(2014) 510-527.

[12] W.X. Wang, Y.L. Shang, L.S. Zhang, Y. Zhang, Global minimization of non-smooth unconstrained problems with filled function, Optim. Lett. 7(2013) 435-446.

[13] F. Wei, Y. Wang, H. Lin, A new filled function method with two parameters for global optimization, J. Optim. Theory. Appl. 163(2014) 510-527.

[14] H. Lin, Y. Wang, L. Fan, A filled function method with one parameter for unconstrained global optimization, Appl. Math. Comput. 218(2011) 3776-3785.

[15] H. Lin, Y. Wang, L. Fan, Y. Gao, A new discrete filled function method for finding global minimize of the integer programming, Appl. Math. Comput. 219(2013) 4371-4378.

[16] T.M. El-Gindy, M.S. Salim, A.I. Ahmed, A new filled function method applied to unconstrained global optimization, Applied Mathematics and Computation 273(2016) 1246-1256.

[17] M.X. Li, Y.L. Shang, G.L. Wang, Modified filled function to solve nonlinear programming problem, Mathematical Computation 4(2015) 50-55.

[18] L.Y. Yuan, Z.P. Wan, Q.H. Tang, Y. Zheng, A class of parameter-free filled functions for box-constrained system of nonlinear equations, Acta Mathematicae Applicatae Sinica, English Series 32(2016) 355-364.

[19] Y.L. Gao, Y.J. Yang, M. You, A new filled function method for global optimization, Applied Mathematics and Computation 268(2015) 685-695.

[20] W.X. Wang, Y.L. Shang, A quasi-filled function approach for nonlinear global integer optimization, in: Proc. 2012 Fifth International Joint Conference on Computational Sciences and Optimization (CSO), 2012.

[21] A. Sahiner, H. Gokkaya, T. Yigit, A new filled function for nonsmooth global optimization, in Proc. AIP Conference Proceeding 1479, 2012.

[22] A. Şahiner, N. Yilmaz, O. Demirozer, Mathematical modeling and an application of the filled function method in entomology, International Journal of Pest Management 60(2014) 232-237.

[23] Z.P. Wan, L.Y. Yuan, J.W. Chen, A filled function method for nonlinear systems of equalities and inequalities, Comput. Appl. Math. 31(2012) 391-405.

[24] X. Wu, K.J. Zhang, C.Y. Sun, Constrained optimal control of switched systems based on modified BFGS algorithm and

filled function method, Journal International Journal of Computer Mathematics 91(2014) 1713-1729.

[25] H.W. Lin, H.R. Li, A new filled function with one parameter to solve global optimization, Open Journal of Optimization 4(2015) 10-20.

[26] F. Branin, Widely convergent methods for finding multiple solutions of simultaneous nonlinear equations, IBM Journal of Research Developments 16(1972) 504-522.

[27] J. Snyman, L. Fatti, A multi-start global minimization algorithm with dynamic search trajectories, Journal of Optimization Theory and Applications 54(1987) 121-141.

[28] A.V. Levy, A. Montalvo, The tunneling algorithm for the global minimization of functions, SIAM Journal on Scientific and Statistical Computing 6(1985) 15-29.

[29] P. Basso, Iterative methods for the localization of the global maximum, SIAM Journal on Numerical Analysis 19(1982) 781-792.

[30] R. Mladineo, An algorithm for finding the global maximum of a multimodal, multivariate function, Math. Program 34(1986) 188-200.

[31] L.Q. Peng, J.Y. Zhang, An entropy weighting mixture model for subspace clustering of high-dimensional data, Pattern Recognition Letters 32(2011) 1154-1161.

[32] Y.P. Wang, A uniform enhancement approach for optimization algorithms: smoothing function method, International Journal of Pattern Recognition and Artificial Intelligence 24(2010) 1111-1131.

[33] Q. Ren, Y.P. Wang, A new hybrid genetic algorithm for job shop scheduling problem, Computer and Operations Research 39(2011) 2291-2299.

[34] C.Y. Dang, Y.B. Sun, Y.P. Wang, Y. Yang, A deterministic annealing algorithm for the minimum concave cost network flow problem, Neural Networks 24(2011) 699-708.

[35] Y.P. Wang, A uniform enhancement approach for optimization algorithms: smoothing function method, International Journal of Pattern Recognition and Artificial Intelligence 24(2010) 1111-1131.

[36] Y.W. Leung, Y.P. Wang, Multiobjective programming using uniform design and genetic algorithm, IEEE Transactions on Systems, Man, and Cybernetics, Part C 30(2000) 293-304.

[37] Z.Y. Wu, F.S. Bai, H.W.J. Lee, Y.J. Yang, A filled function method for constrained global optimization, Journal of Global Optimization 39(2007) 495-507.

[38] L. An, L.S. Zhang, M.L. Chen, A parameter-free filled function for unconstrained global optimization, Journal of Shanghai University (English Edition) 8(2004) 117-123.

[39] Y.L. Shang, P. Li, H.L. Xie, A new parameter-free filled function for unconstrained global optimization, in: Proc. 2010 Third International Joint Conference on Computational Science and Optimization, 2010.

[40] X. Liu, Finding global minima with a computable filled function, Journal of Global Optimization 19(2001) 151-161.

[41] Y. Zhang, L.S. Zhang, Y.T. Xu, New filled functions for nonsmooth global optimization, Applied Mathematical Modelling 33(2009) 3114-3129.

[42] R. Fletcher, Practical Methods of Optimization, Wiley, New York, 1987.

[43] M. Avriel, Nonlinear Programming: Analysis and Methods, Prentice-Hall, Englewood Cliffs, NJ, 1976.